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Navbodh

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MATHEMATICS

B.Sc.Part-III

Numerical Methods and Scientific Computation

(Group-A, First Paper)

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SYLLABUS

MATHEMATICS (M.P.)

B.Sc. PART-III (Group-A, First Paper)

Numerical Methods and Scientific Computation

Unit 1. **Methods for Solving Algebraic and Transcendental Equations :**

Ramanujan, Bisection, Regula Falsi, Secant, Newton-Raphson.

Unit 2. **Interpolation and Numerical Intgration :**

Lagrange interpolation, Finite difference operators.

Interpolation formula using Differences : Gregory-Newton Forward Difference Interpolation, Gregory-Newton Backward Difference Interpolation.

Numerical Integration : Newton-Cote's formulae, Trapezoidal rule, Simpson's 1/3 Rule, Simpson's 3/8 Rule, Gauss Integration.

Unit 3. **Methods to Solve System of Linear Equations :**

Direct method for solving system of linear equations : Gauss elimination, LU decomposition, Cholesky decomposition.

Iterative Method : Jacobi, Gauss-Seidel.

Unit 4. **Numerical Solution of Ordinary Differential Equations :**

Single Step Methods : Picard, Taylor's series, Euler, Runge-Kutta.

Multistep Methods : Predictor-corrector, Modified Euler, Milne-Simpson.

★★★

QUESTIONS

MATHEMATICS (M.P.)

B.Sc. PART-III (First Paper)

Q. No.	Description	Pages
	UNIT 1	
1-25	Short and Long Answer Type Questions	5-23
	Objective Type Questions	24-25
	UNIT 2	
1-34	Short and Long Answer Type Questions	26-53
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	UNIT 3	
1-30	Short and Long Answer Type Questions	57-91
	Objective Type Questions	91-92
	UNIT 4	
1-24	Short and Long Answer Type Questions	93-133
	Objective Type Questions	133-136

UNIT 1

Short and Long Answer Type Questions

Q. 1. Solve $xe^x = 1$ by Ramanujan's method.

Solution : The given equation is $xe^x = 1$

$$\Rightarrow \quad x\left(1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ldots\right)=1$$

$$\Rightarrow \quad x+x^2+\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}+\ldots=1$$

Which is the form of $f(x)=1-(a_1x+a_2x^2+a_3x^3+\ldots\ldots)$

Where $a_1=1, a_2=1, a_3=\frac{1}{2},\ a_4=\frac{1}{6}, a_5=\frac{1}{24},\ldots\ldots$

Hence, by Ramanujan's method,

$$b_1=1, b_2=a_2=1, b_3=a_1b_2+a_2b_1=1+1=2$$

$$b_4=a_1b_3+a_2b_2+a_3b_1=2+1+\frac{1}{2}=\frac{7}{2}$$

$$b_5=a_1b_4+a_2b_3+a_3b_2+a_4b_1$$

$$=\frac{7}{2}+2+\frac{1}{2}+\frac{1}{6}=\frac{37}{6}=6{\cdot}1667$$

$$b_6=a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1$$

$$=\frac{261}{24}=10{\cdot}8750$$

Hence, $b_1/b_2=1,\ b_2/b_3=0{\cdot}5,\ b_3/b_4=0{\cdot}5714$

$b_4/b_5=0{\cdot}5676,\ b_5/b_6=0{\cdot}5670$

Hence, the roots are 0·567. **Ans.**

Q. 2. Find the smallest root of the equation $f(x)=x^3-9x^2+26x-24=0$ by Ramanujan's method.

Solution : $f(x)=1-\frac{26x}{24}+\frac{9}{24}x^2-\frac{1}{24}x^3$

$$=1-\left(\frac{13}{12}x-\frac{3}{8}x^2+\frac{1}{24}x^3\right)$$

Here, $a_1=\frac{13}{12}, a_2=-\frac{3}{8}, a_3=\frac{1}{24}, a_4=a_5=\ldots.=0$

Now, $b_1 = 1, b_2 = a_1 = \frac{13}{12} = 1{\cdot}0833$

$\therefore \quad \frac{b_1}{b_2} = \frac{12}{13} = 0{\cdot}923$

$$b_3 = a_1 b_2 + a_2 b_1 = \frac{13}{12}(1{\cdot}0833) - \frac{3}{8}(1) = 0{\cdot}7986$$

$\Rightarrow \quad \frac{b_2}{b_3} = \frac{1{\cdot}0833}{0{\cdot}7986} = 1{\cdot}356$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1$$

$$= 1{\cdot}0833\,(0{\cdot}7986) + \left(-\frac{3}{8}\right)(1{\cdot}0833) + \frac{1}{24}(1)$$

$$= 0{\cdot}5007$$

Hence $\frac{b_3}{b_4} = \frac{0{\cdot}7986}{0{\cdot}5007} = 1{\cdot}595$

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2$$

$$= 1{\cdot}0833\,(0{\cdot}5007) + \left(-\frac{3}{8}\right)(0{\cdot}7986) + \frac{1}{24}(1{\cdot}0833)$$

$$= 0{\cdot}2880$$

Hence $\frac{b_4}{b_5} = \frac{0{\cdot}50007}{0{\cdot}2880} = 1{\cdot}7382$

Therefore

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3$$

$$= 0{\cdot}1575$$

$\Rightarrow \quad \frac{b_5}{b_6} = 1{\cdot}8286$

$$b_7 = 0{\cdot}0835 \Rightarrow \frac{b_6}{b_7} = 1{\cdot}8862$$

$$b_8 = 0{\cdot}434 \Rightarrow \frac{b_7}{b_8} = 1{\cdot}9240$$

$$b_9 = 0{\cdot}0223 \Rightarrow \frac{b_8}{b_9} = 1{\cdot}9462.$$

The roots of the given equation are 2, 3 and 4 and it can be seen that the successive convergents approach the value 2. **Ans.**

Q. 3. Using Ramanujan's method, find a real roots of the equation

$$1 - x + \frac{x^2}{(\underline{|}2)^2} - \frac{x^3}{(\underline{|}3)^2} + \frac{x^4}{(\underline{|}4)^2} - \ldots\ldots 0.$$

Solution : Let $f(x) = 1 - \left[x - \frac{x^2}{(\underline{|}2)^2} + \frac{x^3}{(\underline{|}3)^2} - \frac{x^4}{(\underline{|}4)^2} + \ldots\ldots\right] = 0$

Hence $a_1 = 1, a_2 = -\frac{1}{(\underline{|}2)^2}, a_3 = \frac{1}{(\underline{|}3)^2}, a_4 = -\frac{1}{(\underline{|}4)^2}, a_5 = \frac{1}{(\underline{|}5)^2}, a_6 = -\frac{1}{(\underline{|}6)^2} \ldots\ldots$

We obtain $b_1 = 1, b_2 = a_1 = 1$

$$b_3 = a_1 b_2 + a_2 b_1 = 1 - \frac{1}{(\underline{|}2)^2} = \frac{3}{4}$$

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1 = \frac{3}{4} - \frac{1}{4} + \frac{1}{36} = \frac{19}{36} = 0{\cdot}5277$$

$$b_5 = \frac{211}{576} = 0{\cdot}3663$$

The successive convergents are :

$$\frac{b_1}{b_2} = 1, \frac{b_2}{b_3} = \frac{4}{3} = 1{\cdot}333$$

$$\frac{b_3}{b_4} = \frac{27}{19} = 1{\cdot}4210, \frac{b_4}{b_5} = 1{\cdot}4408$$

Where the last result is correct to three significant figures. **Ans.**

Q. 4. Find the smallest root, correct to 4 decimal places, of the equation

$$f(x) = 3x - \cos x - 1 = 0.$$

Solution : We have $f(x) = 3x - \cos x - 1$

$$= 1 - 3x + \cos x$$

$$= 1 - 3x + 1 - \frac{x^2}{\underline{|}2} + \frac{x^4}{\underline{|}4} - \frac{x^6}{\underline{|}6} + \ldots\ldots\ldots$$

$$= 2 - 3x - \frac{x^2}{\underline{|}2} + \frac{x^4}{\underline{|}4} - \frac{x^6}{\underline{|}6} + \ldots\ldots\ldots$$

$$= 2\left[1 - \frac{3}{2}x - \frac{x^2}{4} + \frac{x^4}{48} - \frac{x^6}{1440} + \frac{x^8}{80640} - \ldots\ldots\right]$$

Now let $1-\left(\frac{3}{2}x+\frac{x^2}{4}-\frac{x^4}{48}+\frac{x^6}{1440}-\frac{x^8}{80640}\ldots\ldots\right)$

$$=b_1+b_2x+b_3x^2+\ldots\ldots$$

Here, $a_1=\frac{3}{2}, a_2=\frac{1}{4}, a_3=0, a_4=-\frac{1}{48}$

$$a_5=0, a_6=\frac{1}{1440}, a_7=0, a_8=-\frac{1}{80640}\ldots$$

Then we obtain

$b_1=1, b_2=1{\cdot}5, b_3=2{\cdot}5, b_4=4{\cdot}125, b_5=6{\cdot}79167,$

$b_6=11{\cdot}18750, b_7=18{\cdot}42778, b_8=30{\cdot}35365.$

The successive convergents are :

$$\frac{b_1}{b_2}=0{\cdot}66667, \frac{b_2}{b_3}=0{\cdot}60000$$

$$\frac{b_3}{b_4}=0{\cdot}60606, \frac{b_4}{b_5}=0{\cdot}60736$$

$$\frac{b_5}{b_6}=0{\cdot}60708, \frac{b_6}{b_7}=0{\cdot}60710, \frac{b_7}{b_8}=0{\cdot}607102$$

Hence, the required root, correct to four decimal places, is 0·6071. **Ans.**

Q. 5. (a) Find the positive root of the equation $f(x)=x^3-5x-4=0$ using bisection method up to third approximation.

(Indore 2011; Gwalior 13; Bhopal 15; Ujjain 16)

Solution : The equation $f(x)=x^3-5x-4; f(2)=8-10-4<0$

and $f(3)=3^3-15-4>0$. Hence $f(2).f(3)<0$

$\therefore$ The roots lies between 2 and 3.

Again, $I_0=[2,3]$ and $m_1=\frac{2+3}{2}=2{\cdot}5$

lies in interval $I_1=[2{\cdot}5,3]$ because $f(2{\cdot}5).f(3)<0$

On bisecting $m_2=\frac{2{\cdot}5+3}{2}=2{\cdot}75$ and $f(m_2)=f(2{\cdot}75)=3{\cdot}0468$

Now lies in the original interval $I_2=[2{\cdot}5,2{\cdot}75]$ because $f(2{\cdot}5).f(2{\cdot}75)<0$

and $$m_3=\frac{2{\cdot}5+2{\cdot}75}{2}=2{\cdot}625.$$

Hence, approximation roots of the equation up to third approximation is $m_3=2{\cdot}625$

Ans

Q. 5. (b) Find the root of the equation $x^3-x-4=0$ correct to three places between 1 and 2 using bisection Method. (Ujjain 2013, 15; Indore 16; Jabalpur 22)

Solution : $\because f(1)=-\text{ve}$ and $f(2)=+\text{ve}$

$\therefore$ The root lies between 1 and 2.

Now, first approximation is $x_0=\frac{1+2}{2}=\frac{3}{2}=1{\cdot}5$

$$f(x_0)=f(1{\cdot}5)=\frac{27}{8}-\frac{3}{2}-4=-\frac{17}{8}=-\text{ve}$$

$\therefore$ The roots lies between 1·5 and 2.

Second approximation $x_1=\frac{1{\cdot}5+2}{2}=1{\cdot}75$

$$f(x_1)=f(1{\cdot}75)=-0{\cdot}390625=-\text{ve}$$

Hence, the root lies between 1·75 and 2.

Third approximation $x_2=\frac{1{\cdot}75+2}{2}=1{\cdot}875$

and $f(x_2)=f(1{\cdot}875)=0{\cdot}716796=+\text{ve}$

Hence, the root lies between 1·75 and 1·875.

$x_3=\frac{1{\cdot}75+1{\cdot}875}{2}=1{\cdot}8125$ and $f(x_3)=0{\cdot}141845=+\text{ve}$

Hence, the root lies between 1·75 and 1·8125.

$\therefore\ x_4=\frac{1{\cdot}75+1{\cdot}8125}{2}=1{\cdot}78125$

and $f(x_4)=-0{\cdot}129608=-\text{ve}$

Therefore, the root lies between 1·78125 and 1·8125. Continue this process we get the following successive approximation.

$x_5=1{\cdot}796875$ $\quad f(x_5)=+\text{ve}$

$x_6=1{\cdot}7890625$ $\quad f(x_6)=-\text{ve}$

$x_7=1{\cdot}79296875$ $\quad f(x_7)=-\text{ve}$

$x_8=1{\cdot}794921875$ $\quad f(x_8)=-\text{ve}$

$x_9=1{\cdot}7958984375$ $\quad f(x_9)=-\text{ve}$

$x_{10}=1{\cdot}7963865$ $\quad f(x_{10})=+\text{ve}$

$x_{11}=1{\cdot}79614225$

$\because\ x_{10}=x_{11}=1{\cdot}796.$

Hence the accurate value to three places is 1·796. **Ans.**

Q. 5. (c) Find a positive root of the equation x^3-x-1 between 1 and 2 by using bisection method. (Rewa 2014; Chhindwara 22)

Solution : Solve as Q. No. 5(b).

Q. 6. Find a real roots of the equation $x^2 - 5x + 2 = 0$ correct to three places of decimal by using bisection Method. **(Jabalpur 2011, 14, 15)**

Solution : Here $f(x) = x^2 - 5x + 2$ and $f(0) = 2$ and $f(1) = -2$

Therefore a root lies between 0 and 1.

$I_1 = [0,1]$ and $m_1 = \frac{0+1}{2} = 0{\cdot}5$

This interval lies between $[0, 0{\cdot}5]$ because $f(0).f(0{\cdot}5) < 0$

On bisecting, we get $m_2 = \frac{0+0{\cdot}5}{2} = \frac{0{\cdot}5}{2} = 0{\cdot}25$

lies between $[0{\cdot}25, 0{\cdot}5]$ and $m_3 = \frac{0{\cdot}25+0{\cdot}5}{2} = \frac{0{\cdot}75}{2} = 0{\cdot}375$

This lies between $[0{\cdot}375, 0{\cdot}5]$ because $f(0{\cdot}375).f(0{\cdot}5) < 0$

$\therefore \quad m_4 = \frac{0{\cdot}375+0{\cdot}5}{2} = 0{\cdot}4375.$ **Ans.**

Q. 7. Find a real root of the equation $f(x) = x^3 - 2x - 5 = 0$, using bisection method in five stages. **(Indore 2012, 13; Bhopal 13)**

Solution : Let $f(x) = x^3 - 2x - 5$.

Here $f(2) = -1 = -$ve and $f(3) = 16 = +$ve. Therefore, a root lies between 2 and 3.

$\therefore$ The first approximation to the root $x_0 = \frac{1}{2}(2+3) = 2{\cdot}5$

Now $f(x_0) = f(2{\cdot}5) = (2{\cdot}5)^3 - 2(2{\cdot}5) - 5 = 5{\cdot}625 = +$ve.

So the root lies between 2 and x_0 *i.e.*, between 2 and 2·5.

The second approximation to the root $x_1 = \frac{1}{2}(2+2{\cdot}5) = 2{\cdot}25$

Then $f(x_1) = f(2{\cdot}25) = (2{\cdot}25)^3 - 2(2{\cdot}25) - 5 = 1{\cdot}89 = +$ve.

$\therefore$ The root lies between 2 and x_1.

The third approximation to the root $x_2 = \frac{1}{2}(2+2{\cdot}25) = 2{\cdot}125$

Then $f(x_2) = f(2{\cdot}125) = (2{\cdot}125)^3 - 2(2{\cdot}125) - 5 = 3{\cdot}46 = +$ve.

$\therefore$ The root lies between 2 and x_2.

The fourth approximation to the root $x_3 = \frac{1}{2}(2+2{\cdot}125) = 2{\cdot}0625$

Then $f(x_3) = f(2{\cdot}0625) = (2{\cdot}0625)^3 - 2(2{\cdot}065) - 5 = -35 = -$ve.

Hence the root lies between x_2 and x_3.

Hence the fifth approximation to the root $x_4 = \frac{1}{2}(x_2 + x_3) = \frac{1}{2}(2{\cdot}125 + 2{\cdot}0625)$

$= 2{\cdot}09375 = 2{\cdot}094$ correct to 3 places of decimal. **Ans**

Q. 8. (a) Explain Bisection method. **(Jabalpur 2017)**

Ans. Bisection method : The method is based on repeated application of the intermediate value theorem. Suppose that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$ we bisect the interval I_0 at the point $m_1 = \frac{a_0 + b_0}{2}$, thus we get subintervals (a_0, m_1) and (m_1, b_0). We denote I_1 the interval (a_0, m_1) if $f(a_0)f(m_1) < 0$ or the interval (m_1, b_0) if $f(m_1)f(b_0) < 0$. Therefore, I_1 contains the root. We bisect I_1 to get I_2 at whose end points $f(x)$ takes the values of opposite signs and therefore contains the root. The process is repeated p times to get the sequence of subintervals $I_0 \supset I_1 \supset I_2 \ldots\ldots \supset I_p$ such that each subinterval contains the root. We either get the root or find the interval I_p of length $\frac{b-a}{2^p}$ which contains the root. We take the mid point of this last interval to get the desired approximation to the root.

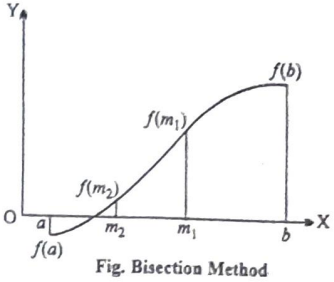


Fig. Bisection Method

In brief, we denote

$$m_{k+1} = \frac{a_k + b_k}{2}$$

and $\quad (a_{k+1}, b_{k+1}) = (a_k, m_{k+1})$, if $f(a_k)f(m_{k+1}) < 0$

or $\quad (m_{k+1}, b_k)$, if $f(m_{k+1})f(b_k) < 0$

It is to be noted that the method uses only the end points of the interval $[a_k, b_k]$ for which $f(a_k)$ and $f(b_k)$ have opposite signs and not the functional values of $f(x)$ at these end points.

Number of iterations required may be obtained from the relation

$$\frac{b_0 - a_0}{2^n} \le \varepsilon$$

or

$$n \ge \frac{\log(b_0 - a_0) - \log \varepsilon}{\log 2}$$

The error decreases linearly with each step by a factor of 0·5. The convergence is slow. To achieve desired accuracy, a large number of iterations are required.

Q. 8. (b) Explain polynomial and Trancedental equations. **(Gwalior 2022)**

Ans. Polynomial : An expression $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n$, where $n > 0$ and $a_0, a_1, a_2, \ldots\ldots, a_n$ is constant is called a polynomial. If $a_n \neq 0$ then its degree is n and if $f(x) = 0$ then it is called polynomial or Algebraic equation.

Trancedental equation : An equation which contains trigonometric function, logerithmic function, exponential function then the equation $f(x) = 0$ is called a trancedental equation.

Example : $a + be^x + c\sin x + d\log x = 0$.

Q. 9. Explain Newton-Raphson Method. **(Jabalpur 2017)**

Ans. The Newton-Raphson formula may be derived analytically by using Taylor Series approximation of the function $f(x)$. If x_k is an approximation to be the root, then

$$f(x_{k+1}) = f(x_k + h) = f(x_k) + hf'(x_k) + \frac{h^2}{2!}f''(x_k) + \ldots = 0 \quad \ldots(1)$$

Neglecting all terms of second and higher degree in h, we obtain

$$f(x_k) + hf'(x_k) = 0$$

$$h = \frac{-f(x_k)}{f'(x_k)} \quad \ldots(2)$$

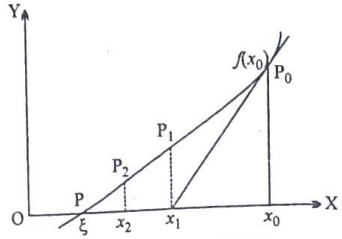


Fig. Newton-Raphson Method.

If $f'(x_k) \neq 0$ for all x_k, then

$$h = (x_{k+1} - x_k) = \frac{-f(x_k)}{f'(x_k)}$$

$$\Rightarrow \qquad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \ldots(3)$$

Consider the curve $f(x)$. Starting from an arbitrary point x_0 as the approximate root of $f(x) = 0$, draw a tangent to the curve at $x = x_0$. The point of intersection of this tangent with x-axis gives the second approximation to the root. Let the point of intersection be x_1. The slope of the tangent is $f'(x_1)$ which equals to $\frac{f(x_0)}{x_0 - x_1}$

$$\frac{f'(x_0)}{x_0 - x_1} = \tan\theta = f'(x_1)$$

or $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In general, $$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

Which is iteration formula and this method is known as Newton-Raphson method.

Q. 10. Find a root of the equation $x^3 - 4x - 9 = 0$ using besection method correct to three decimal places. **(Rewa 2011; Indore 11, 12; Gwalior 14)**

Solution : Let $f(x) = x^3 - 4x - 9$; $f(2) = -9 < 0$, $f(3) = 6 > 0$

$f(2)f(3) < 0$. Thus the root lies between 2 and 3, then first Approximation to the root is

$$m_1 = \frac{2+3}{2} = 2{\cdot}5$$

So the root lies in $[2{\cdot}5, 3]$ because $f(2{\cdot}5)f(3) < 0$

$$m_2 = \frac{2{\cdot}5+3}{2} = 2{\cdot}75$$

So the root lies in $[2{\cdot}5, 2{\cdot}75]$ because $f(2{\cdot}5)f(2{\cdot}75) < 0$

$$m_3 = \frac{2{\cdot}5+2{\cdot}75}{2} = 2{\cdot}625$$

So the root lies in $[2{\cdot}625, 2{\cdot}75]$ because $f(2{\cdot}625)f(2{\cdot}75) < 0$

$$m_4 = \frac{2{\cdot}625+2{\cdot}75}{2} = 2{\cdot}6875$$

So the root lies in $[2{\cdot}6875, 2{\cdot}75]$ because $f(2{\cdot}6875)f(2{\cdot}75) < 0$

$$m_5 = \frac{2{\cdot}6875+2{\cdot}75}{2} = 2{\cdot}7185$$

So the root lies in $[2{\cdot}6875, 2{\cdot}7185]$ because $f(2{\cdot}6875)\ f(2{\cdot}7185) < 0$

$$m_6 = \frac{2{\cdot}6875+2{\cdot}7185}{2} = 2{\cdot}703$$

So the root lies in $[2{\cdot}703, 2{\cdot}7185]$ because $f(2{\cdot}703)f(2{\cdot}7185) < 0$

$$m_7 = \frac{2{\cdot}703+2{\cdot}7185}{2} = 2{\cdot}71075$$

So the root lies in $[2{\cdot}6875, 2{\cdot}71075]$ because $f(2{\cdot}6875)f(2{\cdot}71075) < 0$

$$m_8 = \frac{2{\cdot}6875+2{\cdot}71075}{2} = 2{\cdot}699175.$$

Hence the required root is $2{\cdot}699$. **Ans.**

Q. 11. Compute the real root of $x^3 - 5x + 3 = 0$, in the interval [1, 2] by Regula Falsi method. **(Gwalior 2011; Indore 11; Chhatarpur 22)**

Solution : Here $f(x) = x^3 - 5x + 3$, $f(1) = 1 - 5 + 3 = -1$ and $f(2) = 8 - 10 + 3 = 1$

Therefore, one root lies between 1 and 2 as $f(1).f(2) < 0$.

Using iteration formula,

$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

1st iteration $k = 1$, $\quad x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

Here, $x_0 = 1, x_1 = 2$

$$x_2 = \frac{1\times 1 - 2\times(-1)}{1-(-1)} = \frac{1+2}{2} = \frac{3}{2} = 1{\cdot}5$$

Again $f(x_0)f(x_2) < 0$

Interval $[1,2]$ $f(x_2) = f(1{\cdot}5) = (1{\cdot}5)^3 - 5(1{\cdot}5) + 3 = 3{\cdot}375 - 7{\cdot}5 + 3$

$= -1{\cdot}125$

2nd iteration $k = 2$

Here, $x_1 = 2, x_2 = 1{\cdot}5$ $f(x_1) = 1$, $f(x_2) = -1{\cdot}125$

$$\therefore \qquad x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(-1{\cdot}125) - 1{\cdot}5\times 1}{-1{\cdot}125 - 1}$$

$= 1{\cdot}7647$

Again $f(x_1)f(x_2) < 0$

Interval $[1{\cdot}5, 2]$, $f(x_3) = -0{\cdot}3279$

3rd iteration $k = 3$,

Here, $x_2 = 2, x_3 = 1{\cdot}7647, f(x_2) = 1, f(x_3) = -0{\cdot}3279$

$$\therefore \quad x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{2(-0{\cdot}3279) - 1{\cdot}7647 \times 1}{-0{\cdot}3279 - 1}$$

$$= 1{\cdot}8228$$

Hence $f(x_4) = -0{\cdot}05756$

Here, $f(x_2)f(x_3) \not< 0$, we take $x_2 = 2, x_3 = 1{\cdot}7647$, therefore, the interval in which the root lies becomes $[1{\cdot}7647, 2]$.

Again $f(x_3)f(x_4) \not< 0, x_3 = 2, x_4 = 1{\cdot}8228$

4th iteration, $x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = \frac{2(-0{\cdot}05756) - 1{\cdot}8228(1)}{-0{\cdot}05756 - (-0{\cdot}3279)}$

$= 1{\cdot}8324$. **Ans.**

Q. 12. (a) Find the root of the equation $x^3 - 5x - 7 = 0$ which lies between 2 and 3 by the method of false position. **(Rewa 2014)**

Solution : Let $f(x) = x^3 - 5x - 7$

$f(2) = -9$ and $f(3) = 5$, therefore, one root lies between 2 and 3.

$\therefore$ Taking $x_1 = 2, x_2 = 3, f(x_1) = -9, f(x_2) = 5$,

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) \qquad \ldots(1)$$

$$= 2 - \frac{3-2}{5+9} \times (-9) = 2 + \frac{9}{14} = 2{\cdot}643$$

Now $f(x_3) = f(2{\cdot}643) = -1{\cdot}75246$

Clearly, $f(x_2)f(x_3) < 0$, therefore, the root lies between 2·643 and 3.

In eqn. (1) taking $x_1 = 2{\cdot}643, x_2 = 3, f(x_1) = -1{\cdot}75246$,

$$x_4 = 2{\cdot}643 - \frac{3 - 2{\cdot}643}{5 + 1{\cdot}75246}(1{\cdot}75246) = 2{\cdot}736$$

Now $f(x_4) = f(2{\cdot}736) = -0{\cdot}199$

Clearly, $f(x_2)f(x_4) < 0$, so, the root lies between 2·736 and 3.

In eqn. (1) taking $x_1 = 2{\cdot}736, x_2 = 3, f(x_1) = -0{\cdot}1999$,

$$x_5 = 2{\cdot}736 - \frac{3 - 2{\cdot}736}{5 + 0{\cdot}199} \times (-0{\cdot}199)$$

$$= 2{\cdot}736 + \frac{0{\cdot}264}{5{\cdot}199} \times 0{\cdot}199 = 2{\cdot}746$$

Now $f(x_5) = f(2{\cdot}746) = -0{\cdot}024$

Clearly, $f(x_2)f(x_5) < 0$

Therefore, the root lies between 2·746 and 3.

In eqn. (1), taking $x_1 = 2{\cdot}746, x_2 = 3, f(x_1) = -0{\cdot}024$,

$$x_6 = 2{\cdot}746 - \frac{3 - 2{\cdot}746}{5 + 0{\cdot}024}(-0{\cdot}024) = 2{\cdot}747$$

Now $f(x_6) = f(2{\cdot}747) = -0{\cdot}006$

Clearly, $f(x_2)f(x_6) < 0$, so the root lies between 2·747 and 3.

In eqn. (1), taking $x_1 = 2{\cdot}747, x_2 = 3, f(x_1) = -0{\cdot}006$

$$x_7 = 2{\cdot}747 - \frac{3 - 2{\cdot}747}{5 + 0{\cdot}006}(-0{\cdot}006) = 2{\cdot}747$$

Hence, the root is 2·747 correct to 3 decimal places. **Ans.**

Q. 12. (b) Find the root of the equation $x^3 - 5x - 5 = 0$ which lies between 2 and 3 by the method of false position. **(Ujjain 2022)**

Solution : Solve as Q. No. 12 (a).

Q. 13. Find the least positive root of the equation $x^4 - x - 10 = 0$ using Regula Falsi method up to third approximation. **(Indore 2011, 17)**

Solution : Given $x^4 - x - 10 = 0$

Since $f(0) = -10, f(1) = -10, f(2) = 4$

$\therefore$ Least positive root lies in the interval (1, 2).

By Regula Falsi method, we have

Approximation formula $x_{k+1} = \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$

At $k = 1$, $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$$= \frac{1f(2) - 2f(1)}{f(2) - f(1)} = \frac{1 \times 4 - 2 \times (-10)}{4 - (-10)} = 1{\cdot}7143$$

Now, $f(x_2) = f(1{\cdot}7143)$

$= -3{\cdot}0776$

For second approximation putting $k = 2$, we get

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}, \qquad [\text{where } x_1 = 2, x_2 = 1{\cdot}7143]$$

$$= \frac{2 \times (-3{\cdot}0776) - 1{\cdot}7143 \times (4)}{-3{\cdot}0776 - 4} = 1{\cdot}8385$$

Now, $f(x_3) = f(1{\cdot}8385) = -0{\cdot}4135$

$\therefore$ The root lies between 1·8385 and 2. For third approximation putting $k = 3$, we get

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}, \qquad [\text{where } x_2 = 2, x_3 = 1{\cdot}8385]$$

$$= \frac{2(-0{\cdot}4135) - 1{\cdot}8385(4)}{-0{\cdot}4135 - 4} = 1{\cdot}8536. \quad \textbf{Ans.}$$

Q. 14. Find a real root of the equation $x^3 - 9x + 1 = 0$ by the method of false position. **(Bhopal 2022)**

Solution : Let $f(x) = x^3 - 9x + 1$

Here $f(2) = -9, f(3) = 1$

$\therefore$ Root lies between 2 and 3.

Taking $x_1 = 2, x_2 = 3 \Rightarrow f(x_1) = -9, f(x_2) = 1$

The method of false position, gives

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) \qquad ...(1)$$

$$\Rightarrow \qquad x_3 = 2 - \frac{3-2}{1+9}(-9) = 2{\cdot}9$$

$$f(x_3) = f(2{\cdot}9) = (2{\cdot}9)^3 - 9(2{\cdot}9) + 1 = -0{\cdot}711$$

Clearly, $f(x_2) f(x_3) < 0$, therefore, the root lies between 2·9 and 3.

In eqn. (1), taking $x_1 = 2{\cdot}9, x_2 = 3, f(x_1) = -0{\cdot}711, f(x_2) = 1$

$$x_4 = 2{\cdot}9 - \frac{3 - 2{\cdot}9}{1 + 0{\cdot}711}(-0{\cdot}711) = 2{\cdot}9416$$

Hence $f(x_4) = (2{\cdot}9416)^3 - 9(2{\cdot}9416) + 1 = -0{\cdot}0207$

Clearly, $f(x_2) f(x_4) < 0$, so, the root lies between 2·9416 and 3.

Hence in eqn. (1), taking $x_1 = 2{\cdot}9416, x_2 = 3, f(x_1) = -0{\cdot}0207, f(x_2) = 1$

$$x_5 = 2{\cdot}9416 - \frac{0{\cdot}0584}{1{\cdot}0207}(-0{\cdot}0207) = 2{\cdot}9428$$

$f(x_5) = -0{\cdot}0003$,

$$x_6 = 2{\cdot}9428 - \frac{0{\cdot}0572}{1{\cdot}0003}(-0{\cdot}0003) = 2{\cdot}9428$$

Hence, the root is 2·942 correct to three decimal places. **Ans.**

Q. 15. Apply regula-falsi method to solve the equation $3x - \cos x - 1 = 0$. **(Rewa 2017)**

Solution : Let $f(x) = 3x - \cos x - 1$

Here $f(0{\cdot}60) = 0{\cdot}02534, f(0{\cdot}61) = 0{\cdot}01035$

$\therefore$ One root lies between 0·60 and 0·61.

From method of false position.

First iteration : $x_1 = \dfrac{0{\cdot}60 f(0{\cdot}61) - 8{\cdot}61 f(0{\cdot}60)}{f(0{\cdot}61) - f(0{\cdot}60)} = 0{\cdot}60710$

and $f(0{\cdot}60710) = -0{\cdot}00001$ so one root lies between 0·60710 and 0·61.

Second iteration : $x_2 = \dfrac{0{\cdot}60710 f(0{\cdot}61) - 0{\cdot}61 f(0{\cdot}60710)}{f(0{\cdot}61) - f(0{\cdot}60710)}$

$= 0{\cdot}60710$

Hence, the root correct to four decimal places is 0·6071. **Ans.**

Q. 16. Determine the root of $xe^x - 2 = 0$ by method of false position. (Ujjain 2013)

Solution : Let $f(x) = xe^x - 2 = 0$ so that $f(0{\cdot}8) = -0{\cdot}21957$ and $f(0{\cdot}9) = 0{\cdot}21364$. Therefore, one root lies between 0·8 and 0·9.

$\therefore$ Taking $x_1 = 0{\cdot}8, x_2 = 0{\cdot}9, f(x_1) = -0{\cdot}21957$ and $f(x_2) = 0{\cdot}21364$ in the method of false position, we get

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) \qquad ...(1)$$

$$= 0{\cdot}8 - \frac{0{\cdot}9 - 0{\cdot}8}{0{\cdot}21364 - (-0{\cdot}21957)}(-0{\cdot}21957)$$

$$= 0{\cdot}8 - \frac{0{\cdot}1}{0{\cdot}43321}(-0{\cdot}21957) = 0{\cdot}85068.$$

Now, $f(x_3) = f(0{\cdot}85068) = -0{\cdot}00836$. Clearly, $f(x_2) f(x_3) < 0$ so the root lies between 0·85068 and 0·9.

Taking $x_1 = f(0{\cdot}85068) = -0{\cdot}00836$ in eqn. (1), the better approximation x_4 is given by

$$x_4 = 0{\cdot}85068 - \frac{0{\cdot}9 - 0{\cdot}85068}{0{\cdot}21364 - (-0{\cdot}00836)}(-0{\cdot}00836)$$

$$= 0{\cdot}85068 - \frac{0{\cdot}04932}{0{\cdot}222} \times (-0{\cdot}00836) = 0{\cdot}85254.$$

Now $f(x_4) = -0{\cdot}00028$. Clearly, $f(x_2) f(x_4) < 0$.

So, the root lies between 0·85254 and 0·9.

Taking $x_1 = 0{\cdot}85254, x_2 = 0{\cdot}9, f(x_1) = -0{\cdot}00028$ in eqn. (1), the better approximation x_5 is given by

$$x_5 = 0{\cdot}85254 - \frac{0{\cdot}9 - 0{\cdot}85254}{0{\cdot}21364 - (-0{\cdot}00028)}(-0{\cdot}00028)$$

$$= 0{\cdot}85254 - \frac{0{\cdot}04746}{0{\cdot}21392}(-0{\cdot}00028) = 0{\cdot}85260.$$

Now $f(x_5) = f(0{\cdot}85260) = -0{\cdot}00002$. Clearly, $f(x_2) f(x_5) < 0$ so the root lies between 0·85260 and 0·9.

Taking $x_1 = 0{\cdot}85260, x_2 = 0{\cdot}9, f(x_1) = -0{\cdot}00002$ in eqn. (1), the better approximation x_6 is given by

$$x_6 = 0{\cdot}85260 - \frac{0{\cdot}9 - 0{\cdot}85260}{0{\cdot}21364 - (-0{\cdot}00002)}(-0{\cdot}00002)$$

$$= 0{\cdot}85260 - \frac{0{\cdot}0474}{0{\cdot}21366}(-0{\cdot}00002) = 0{\cdot}85260.$$

Since $x_6 = x_5$, the real root of the given equation correct to five decimal places = 0·85260

Ans.

Q. 17. (a) Solve by Newton -Raphson Method : $x^3 - 3x + 1 = 0$.
(Ujjain 2011; Indore 13, 16; Jabalpur 15; Gwalior 16; Rewa 22; Ujjain 22)

Solution : Let $f(x) = x^3 - 3x + 1$

$\therefore \quad f'(x) = 3x^2 - 3$

We have $f(0) = 1 = +ve$ and $f(1) = -1 = -ve$.
Therefore, the roots lie between 0 and 1.

Taking $x_0 = \frac{0+1}{2} = 0{\cdot}5$, now by Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} \qquad ...(1)$$

Putting $n = 0, 1, 2, 3, ...$successively in eqn. (1) we have

$$x_1 = x_0 - \frac{x_0^3 - 3x_0 + 1}{3x_0^2 - 3} = 0{\cdot}5 - \frac{(-0{\cdot}375)}{(-2{\cdot}25)} = 0{\cdot}3333$$

$$x_2 = x_1 - \frac{x_1^3 - 3x_1 + 1}{3x_1^2 - 3} = 0{\cdot}3333 - \frac{(0{\cdot}0370459)}{(-2{\cdot}666733)} = 0{\cdot}3472$$

$$x_3 = x_2 - \frac{x_2^3 - 3x_2 + 1}{3x_2^2 - 3} = 0{\cdot}3472 - \frac{(0{\cdot}00025)}{(-2{\cdot}638356)} = 0{\cdot}3473$$

Since $x_2 = x_3 = 0{\cdot}347$, therefore the required root = 0·347. **Ans.**

Q. 17. (b) Solve by Newton-Raphson method : $x^3 - 5x + 3 = 0$. **(Rewa 2022)**

Solution : Solve as Q. No. 17(a).

Q. 18. (a) Evaluate $(12)^{1/3}$ using Newton-Raphson method. **(Indore 2012)**

Solution : Since $2^3 = 8$ and $3^3 = 27$
Therefore the value of $(12)^{1/3}$ lies between 2 and 3.

Let $(12)^{1/3} = x \Rightarrow x^3 = 12 \Rightarrow x^3 - 12 = 0$

$\Rightarrow \quad f(x) = x^3 - 12$

$\Rightarrow \quad f'(x) = 3x^2$

Taking $x_0 = 2{\cdot}5$
By Newton-Raphson formula,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Putting $k = 0$, the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2{\cdot}5 - \frac{(2{\cdot}5)^3 - 12}{3(2{\cdot}5)^2}$$

$$= 2{\cdot}3065$$

Putting $k = 1$, the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2{\cdot}3065 - \frac{(2{\cdot}3065)^3 - 12}{3(2{\cdot}3065)^2}$$

$$= 2{\cdot}2896$$

Putting $k = 2$, the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2{\cdot}2896 - \frac{(2{\cdot}2896)^3 - 12}{3(2{\cdot}2896)^2}$$

$$= 2{\cdot}2896 - \frac{0{\cdot}002697}{15{\cdot}7268} = 2{\cdot}289411. \qquad \textbf{Ans.}$$

Q 18. (b) Find a root of the equation $x^3 - 3x - 5 = 0$ correct to three places of decimal by using Newton-Raphson method. (Jabalpur 2011; Indore 17; Bhopal 22)

Solution : Let $f(x) = x^3 - 3x - 5$

$\Rightarrow \quad f'(x) = 3x^2 - 3$

Clearly a root lies between 2 and 3.

Since $f(2) = -3$ and $f(3) = 13$

We choose $x_0 = 3$

Newton's formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3}$

For $n = 0$ the first approximation x_1 is given by

$$x_1 = x_0 - \frac{x_0^3 - 3x_0 - 5}{3x_0^2 - 3} = 3 - \frac{(3)^3 - 3(3) - 5}{3(3)^2 - 3} = 2{\cdot}46$$

For $n = 1$ the second approximation x_2 is given by

$$x_2 = x_1 - \frac{x_1^3 - 3x_1 - 5}{3x_1^2 - 3} = 2{\cdot}46 - \frac{(2{\cdot}46)^3 - 3(2{\cdot}46) - 5}{3(2{\cdot}46)^2 - 3} = 2{\cdot}295$$

For $n = 2$ the third approximation x_3 is given by

$$x_3 = x_2 - \frac{x_2^3 - 3x_2 - 5}{3x_2^2 - 3} = 2{\cdot}295 - \frac{(2{\cdot}295)^3 - 3(2{\cdot}295) - 5}{3(2{\cdot}295)^2 - 3}$$

$$= 2{\cdot}279 \text{ (upto 3 places of decimals)}$$

For $n = 3$ the fourth approximation x_4 is given by

$$x_4 = x_3 - \frac{x_3^3 - 3x_3 - 5}{3x_3^2 - 3} = 2{\cdot}279 - \frac{(2{\cdot}279)^3 - 3(2{\cdot}279) - 5}{3(2{\cdot}279)^2 - 3}$$

$$= 2{\cdot}279 \text{ (upto 3 places of decimals)}$$

Since $x_3 = x_4$, the real root of the given equation correct to three decimal places is 2·279. **Ans.**

Q. 19. Find the root of $x^2 - 5x + 2 = 0$ correct to five decimal places, by Newton-Raphson method. **(Ujjain 2014; Bhopal 16; Jabalpur 22)**

Solution : Let $f(x) = x^2 - 5x + 2$, $f'(x) = 2x - 5$

By Newton-Raphson formula we have

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - 5x_0 + 2}{(2x_0 - 5)} = \frac{x_0^2 - 2}{2x_0 - 5}$$

Taking $x_0 = 0$ the first approximation is

$$x_1 = \frac{-2}{-5} = 0{\cdot}4$$

Again putting $n = 0$ and $x_1 = 0{\cdot}4$, we get the second approximation is

$$x_2 = \frac{x_1^2 - 2}{2x_1 - 5} = \frac{(0{\cdot}4)^2 - 2}{2(0{\cdot}4) - 5} = 0{\cdot}438095$$

$$x_3 = \frac{x_2^2 - 2}{2x_2 - 5} = \frac{(0{\cdot}438095)^2 - 2}{2(0{\cdot}438095) - 5} = 0{\cdot}438447$$

$$x_4 = \frac{x_3^2 - 2}{2x_3 - 5} = \frac{(0{\cdot}438447)^2 - 2}{2(0{\cdot}438447) - 5} = 0{\cdot}438447$$

Clearly, $x_3 = x_4$ so that real root is 0·438447. **Ans.**

Q. 20. Prove that Newton's method has quadratic convergence.
(Gwalior 2011, 13, 17)

Solution : We have from Newton-Raphson approximation formula,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad ...(1)$$

Suppose x_k differs from the root ξ by a small quantity ε_k so that $x_k = \xi + \varepsilon_k$ and $x_{k+1} = \xi + \varepsilon_{k+1}$.

From eqn. (1), we have

$$\xi + \varepsilon_{k+1} = \xi + \varepsilon_k - \frac{f(\xi + \varepsilon_k)}{f'(\xi + \varepsilon_k)}$$

$$\Rightarrow \qquad \varepsilon_{k+1} = \varepsilon_k - \frac{f(\xi + \varepsilon_k)}{f'(\xi + \varepsilon_k)}$$

By Taylor's expansion in *nbd* of ξ to the function $f(\xi + \varepsilon_k)$ and $f'(\xi + \varepsilon_k)$, we get

$$\varepsilon_{k+1} = \frac{\varepsilon_k\left[f'(\xi) + \varepsilon_k f'(\xi) + \frac{1}{2!}\varepsilon_k^2 f''(\xi) + ...\right] - \left[f(\xi) + \varepsilon_k f'(\xi) + \frac{\varepsilon_k^2}{2!} f''(\xi) + ...\right]}{f'(\xi) + \varepsilon_k f''(\xi) + \frac{\varepsilon_k^2}{2!} f'''(\xi) + ...}$$

Since ξ is a root so that $f(\xi) = 0$,

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}\varepsilon_k^2 + 0(\varepsilon_k^3)$$

Neglecting ε_k^3 and higher order, we get

$$\varepsilon_{k+1} = C\varepsilon_k^2$$

Where, $C = \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

Thus Newton-Raphson method has second order convergence. **Proved.**

Q. 21. By using Newton-Raphson's method find the positive root of $x^4 - x - 10 = 0$ which is nearer to 2, correct to three places of decimal.
(Gwalior 2012; Bhopal 14, 17; Ujjain 16; Jabalpur 16, 17)

Solution : Let $f(x) = x^4 - x - 10$,

$\therefore f'(x) = 4x^3 - 1$

By Newton formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - x_n - 10}{4x_n^3 - 1} = \frac{3x_n^4 + 1}{4x_n^3 - 1}$$

We have $f(1{\cdot}8) = -1{\cdot}3024 = -$ ve, $f(2) = 4 = +$ ve, therefore the root lies between 1·8 and 2 we may consider the approximation value of the root as 2.

$\therefore$ Taking $x_0 = 2, n = 0$ in eqn. (1) the first approximation is given by

$$x_1 = \frac{3x_0^4 + 10}{4x_0^3 - 1} = \frac{3(2^4) + 10}{4(2)^3 - 1} = \frac{58}{31} = 1{\cdot}871$$

For $n = 1$, the second approximation x_2 is

$$x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1} = \frac{3(1{\cdot}871)^4 + 10}{4(1{\cdot}871)^3 - 1} = 1{\cdot}856$$

For $n = 2$, the third approximation x_3 is

$$x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3(1{\cdot}856)^4 + 10}{4(1{\cdot}856)^3 - 1} = 1{\cdot}856$$

Since $x_2 = x_3$, the required root is 1·856. **Ans.**

Q. 22. Derive secant method. How this method is different from the false position method ? **(Jabalpur 2011; Gwalior 13; Ujjain 14)**

Solution : In secant method the equation $f(x) = 0$ is replaced as a straight line or chord, which is joining the points $(x_k, f(x_k))$ and $(x_{k-1}, f(x_{k-1}))$. The approximate root is the intersecting of the chord and X-axis.

The equation of a chord joining the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$ is

$$\frac{y - f(x_{k-1})}{x - x_{k-1}} = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \quad ...(1)$$

Taking intersection of eqn. (1) with X- axis, we get

$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})} \quad ...(2)$$

Which is an approximation to the root.

To find next approximation x_{k+2} replace x_{k-1} by x_k and x_k by x_{k+1}. This method is different from the false position method in the following reasons :

(i) Here $f(x_{k-1})f(x_k) < 0$ is not the required condition.

(ii) If at any iteration $f(x_k) = f(x_{k-1})$ this method fails and show that it does not converge necessarily. This is a drawback of secant method over the method of false position which always converges.

Q. 23. Find the least positive root of the equation $x^3 - x - 1 = 0$ by secant method. (Rewa 2011)

Solution : Given, $f(x) = x^3 - x - 1$

Since $f(1) = -1, f(2) = 5$

Hence root lies in the interval (1, 2).

By iteration formula, we have

$$x_{k+1} = \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}$$

First iteration $k = 1,\ x_{k-1} = x_0 = 1,\ x_k = x_1 = 2$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{1 \times 5 - 2 \times (-1)}{5 + 1} = \frac{7}{6} = 1{\cdot}1666,$$

$$f(x_2) = f(1{\cdot}1666) = -0{\cdot}5789$$

Second iteration $k = 2,\ x_{k-1} = x_1 = 2, x_k = x_2 = 1{\cdot}1666$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(-0{\cdot}5789) - 1{\cdot}1666 \times 5}{-0{\cdot}5789 - 5}$$

$$= 1{\cdot}2531,\ f(x_3) = f(1{\cdot}2531) = -0{\cdot}2854$$

Third iteration $k = 3,\ x_{k-1} = x_2 = 1{\cdot}1666,\ x_k = x_3 = 1{\cdot}2531$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = \frac{1{\cdot}1666(-0{\cdot}2854) - 1{\cdot}2531(-0{\cdot}5789)}{-0{\cdot}2854 - (-0{\cdot}5789)}$$

$$= 1{\cdot}3371$$

Similarly, $x_5 = 1{\cdot}325$. **Ans.**

Q. 24. Use the secant method to estimate the root of the equation $x^2 - 4x - 10 = 0$ with the initial estimates of $x_1 = 4$ and $x_2 = 2$. (Ujjain 2017)

Solution : Here $f(x) = x^2 - 4x - 10, f(4) = 16 - 16 - 10 = -10 < 0$

$$f(2) = 4 - 8 - 10 = -14 < 0$$

By secant method $x_{n+2} = \frac{x_{n+1} f(x_n) - x_n f(x_{n+1})}{f(x_n) - f(x_{n+1})}$

On putting $n = 1$ $\quad x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$

$$= \frac{2 \times (-10) - 4 \times (-14)}{-10 - (-14)}$$

$$= \frac{-20 + 56}{4} = \frac{36}{4} = 9$$

$$f(x_3) = f(9) = 35$$

On putting $n = 2$ $\quad x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)}$

$$= \frac{9 \times (-14) - 2 \times 35}{-14 - 35} = 4$$

$$f(x_4) = f(4) = -10$$

On putting $n = 3$ $\quad x_5 = \frac{x_4 f(x_3) - x_3 f(x_4)}{f(x_3) - f(x_4)}$

$$= \frac{4 \times 35 - 9 \times (-10)}{35 + 10} = 5{\cdot}111$$

Similarly, $\quad x_6 = 5{\cdot}9565 \quad (n = 4)$

$x_7 = 5{\cdot}7275 \quad (n = 5)$

$x_8 = 5{\cdot}7411 \quad (n = 6)$

$x_9 = 5{\cdot}7416 \quad (n = 7)$

Here $\quad x_8 = x_9 = 5{\cdot}741.$

Hence the desired root is $5{\cdot}741$ correct to three decimal places. **Ans.**

Q. 25. Use the secant method to find the root of the equation $x^3 - 3x + 1 = 0$ with initial estimates of $x_0 = 1$ and $x_1 = 0{\cdot}5$ only solve till three iteration.

Solution : Here $x_0 = 1, x_1 = 0{\cdot}5, f(x) = x^3 - 3x + 1 = 0$

$\Rightarrow f(x_0) = -1$ and $f(x_1) = -0{\cdot}375$

By secant method, $x_{n+1} = \frac{x_{n-1} f(x_n) - x_n \cdot f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

For $n = 1$, the first iteration is

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \frac{(1)(-0{\cdot}375) - 0{\cdot}5 \times (-1)}{-0{\cdot}375 - (-1)} = 0{\cdot}2$$

$$f(x_2) = f(0{\cdot}2) = (0{\cdot}2)^3 - 3 \times (0{\cdot}2) + 1 = 0{\cdot}408$$

For $n = 2$, the second iteration is

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{0{\cdot}5 \times 0{\cdot}408 - 0{\cdot}2 \times (-0{\cdot}375)}{0{\cdot}408 - (-0{\cdot}375)}$$

$$= 0{\cdot}356322$$

$$f(x_3) = (0{\cdot}356322)^3 - 3(0{\cdot}356322) + 1$$

$$= -0{\cdot}023722.$$

For $n = 3$, the third iteration is

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0{\cdot}2 \times (-0{\cdot}023722) - 0{\cdot}356322 \times 0{\cdot}408}{-0{\cdot}023722 - 0{\cdot}408}$$

$$= 0{\cdot}347733.$$

Ans.

Objective Type Questions

1. Approximation root of $f(x)=0$ in bisection method is : **(Indore 2017)**

(a) $\frac{x_1+x_2}{2}$, if $f(x_1).f(x_2)<0$ (b) $\frac{x_1+x_2}{2}$, if $f(x_1).f(x_2)>0$

(c) $\frac{x_1+x_2}{2^2}$, if $f(x_1).f(x_2)<0$ (d) None of these.

2. Approximation to the root of $f(x)=0$ in case of $f(x_0).f(x_1)<0$ by Regula Falsi method :

(a) $x=x_0-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)$ (b) $x=x_1-\frac{x_1-x_0}{f(x_1)-f(x_0)}f(x_0)$

(c) $x=x_0-\frac{f(x_1)-f(x_0)}{x_1-x_0}f(x_0)$ (d) None of these.

3. The first approximation to the root of $x^3-4x-9=0$ using bisection method is : **(Jabalpur 2014, 22; Indore 16; Bhopal 16)**

(a) 3·5 (b) 2 (c) 2·5 (d) None of these.

4. If x_n is the n^{th} approximation to the root of $f(x)=0$ in Newton-Raphson method, then x_{n+1} is equal to :

(a) $x_n-\frac{f(x_n)}{f'(x_n)}$ (b) $x_n-\frac{f'(x_n)}{f(x_n)}$ (c) $x_{n-1}-\frac{f'(x_n)}{f(x_n)}$ (d) None of these.

5. Newton's iterative formula to find $N^{\frac{1}{2}}$ is :

(a) $x_{n+1}=x_n+\frac{N}{x_n}$ (b) $x_{n+1}=\frac{1}{2}\left(x_n+\frac{N}{x_n}\right)$

(c) $x_{n+1}=\frac{1}{2}\left(x_n+\frac{N^{1/2}}{x_n}\right)$ (d) None of these.

6. The order of convergence in Newton-Raphson method is : **(Jabalpur 2015, 16, 17; Bhopal 17; Chhatarpur 22; Ujjain 22)**

(a) 0 (b) 3 (c) 2 (d) 1.

7. If the root of $f(x)=0$ is in $[a, b]$, then $f(a).f(b)$ is : **(Bhopal 2015)**

(a) Zero (b) Less than zero

(c) Greater than zero (d) None of these.

8. If $\sqrt{2}$ and $1+i$ are the roots of the biquadratic equation, then remaining two roots are :

(a) $\sqrt{2}, 1-i$ (b) $-\sqrt{2}, 1+i$ (c) $\sqrt{2}, 1+i$ (d) $-\sqrt{2}, 1-i$.

9. Two roots of cubic equation are 2, 1+ i, then that equation is : **(Jabalpur 2012)**

(a) x^3+4x^2+6x-4 (b) x^3-4x^2+6x-4

(c) x^3+4x^2-6x-4 (d) x^3-4x^2-6x+4.

10. If the changes of signs in $f(x)$ is n, then $f(x)=0$ has :

(a) Number of positive root is less than n

(b) Number of negative root is more than n

(c) Number of roots is exactly n

(d) None of these.

11. The number of real roots in odd degree equation is :

(a) At least one (b) More than 1 (c) Not any one (d) None of these.

12. In secant method the formula to find the root of $f(x)=0$: **(Rewa 2011; Indore 16)**

(a) $x_{n+1}=x_n-\frac{(x_n-x_{n-1})f(x_n)}{f(x_n)-f(x_{n-1})}$

(b) $x_{n+1}=x_n+\frac{(x_n-x_{n-1})f(x_n)}{f(x_n)-f(x_{n-1})}$

(c) $x_{n+1}=x_n-\frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}f(x_n)$

(d) $x_{n+1}=x_n-\frac{f(x_n)-f(x_{n-1})}{(x_n-x_{n-1})f(x_n)}$.

13. Newton - Raphson method is : **(Bhopal 2011)**

(a) $x_{k+1}=\frac{x_{k-1}f(x_k)-x_kf(x_{k-1})}{f(x_k)-f(x_{k-1})}$

(b) $x_{k+1}=x_{k-1}-\frac{f(x_{k-1})}{f(x_k)-f(x_{k-1})}(x_k-x_{k-1})$

(c) $x_k=x_{k-1}-\frac{f(x_k)}{f'(x_k)}$

(d) $x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$.

14. The convergence in bisection method is :

(a) Linear (b) Quadratic (c) Very fast (d) Very slow.

15. The root of the equation $xe^x=1$ by Ramanujan's method :

(a) 0·567 (b) 1·56 (c) 0·056 (d) None of these.

16. The second approximation root of the equation x^2-2x-5 by bisection method :

(a) 2·15 (b) 2·25 (c) 2·215 (d) None of these.

17. By Regula-Falsi method, the first approximate value of real root of x^3-5x+3 in the interval [1, 2] is :

(a) 1·5 (b) 1·7 (c) 1·75 (d) 1·752.

18. The first approximation of the root of $x^3-x-1=0$ using bisection method is :

(a) 1 (b) 2·5 (c) 1·5 (d) None of these

Ans. 1. (a), 2. (a), 3. (c), 4. (a), 5. (b), 6. (c), 7. (b), 8. (d), 9. (b), 10. (b), 11. (a), 12. (a), 13. (d), 14. (d), 15 (a). 16. (b), 17. (a), 18. (c).

UNIT 2

Short and Long Answer Type Questions

Q. 1. Define interpolation and extrapolation. (Gwalior 2013; Ujjain 14; Rewa 22)

Ans. Suppose we are given the following values of $y = f(x)$ for a set of values of x.

$$x : x_1 \quad x_2 \ldots\ldots x_n$$
$$y : y_1 \quad y_2 \ldots\ldots y_n$$

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_1 and x_n is called interpolation. Thus interpolation is the technique of estimating the value of function for any intermediate value of the independent variable while the process of computing the value of the function outside the given range is called extrapolation.

Definition : (i) Interpolation : An operation of estimating the values of a function for any intermediate value of the argument when the value of the function corresponding to a number of the values of argument are given, is called interpolation.

(ii) Extrapolation : It is the operation estimating the values of function outside the given range of the values of argument.

Example : Given,

x	4	5	7	10	11
y	48	100	294	900	1210

Then estimate the value of y for $x = 6$, 8 or 9 etc. *i.e.*, to found the value of y within the given range of x (from $x = 4$ to $x = 11$) is called interpolation. On the other hand if we estimate the value of y outside the range of x *i.e.*, for $x = 3$ or $x = 12$, then it is called extrapolation.

Q. 2. (a) Show that Lagrange's formula can be expressed in the form :

$$P_n(x) = \sum_{r=0}^{n} \frac{\phi(x) f(x_r)}{(x - x_r)\phi'(x_r)}, \text{ where } \phi(x) = \prod_{r=0}^{n} (x - x_r) .$$

Solution : Given, $\phi(x) = \prod_{r=0}^{n} (x - x_r)$

$$\Rightarrow \quad \phi(x) = (x - x_0)(x - x_1)\ldots(x - x_{r-1})(x - x_r)(x - x_{r+1})\ldots(x - x_n)$$

Taking logarithm on both the sides, we have

$$\log[\phi(x)] = \log[(x - x_0)(x - x_1)\ldots(x - x_{r-1})(x - x_r)(x - x_{r+1})\ldots(x - x_n)]$$

or $$\log \phi(x) = \log(x - x_0) + \log(x - x_1) + \ldots\ldots + \log(x - x_{r-1}) + \log(x - x_r) + \log(x - x_{r+1}) + \ldots\ldots + \log(x - x_n)$$

$$\therefore \frac{d}{dx}[\log \phi(x)] = \frac{1}{x - x_0} + \frac{1}{x - x_1} + \ldots\ldots + \frac{1}{x - x_{r-1}} + \frac{1}{x - x_r} + \frac{1}{x - x_{r+1}} + \ldots\ldots + \frac{1}{x - x_n}$$

$$\Rightarrow \quad \frac{\phi'(x)}{\phi(x)} = \frac{1}{x - x_0} + \frac{1}{x - x_1} + \ldots + \frac{1}{x - x_{r-1}} + \frac{1}{x - x_r} + \frac{1}{x - x_{r+1}} + \ldots + \frac{1}{x - x_n}$$

$$\therefore \quad \phi'(r) = (x_r - x_0)(x_r - x_1)\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots\ldots(x_r - x_n)$$

Now Lagrange's formula is

$$P_n(x) = \frac{(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n)} f(x_1) + \ldots\ldots$$

$$+ \frac{(x - x_0)\ldots\ldots(x - x_{r-1})(x - x_{r+1})\ldots\ldots(x - x_n)}{(x_r - x_0)\ldots\ldots(x_r - x_{r-1})(x_r - x_{r+1})\ldots\ldots(x_r - x_n)} f(x_r) + \ldots\ldots$$

$$+ \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})} f(x_n)$$

$$= \frac{\phi(x)}{(x - x_0)} \frac{f(x_0)}{\phi'(x_0)} + \frac{\phi(x)}{(x - x_1)} \frac{f(x_1)}{\phi'(x_1)} + \ldots\ldots + \frac{\phi(x)}{(x - x_r)} \frac{f(x_r)}{\phi'(x_r)} + \ldots\ldots + \frac{\phi(x)}{(x - x_n)} \frac{f(x_n)}{\phi'(x_n)}$$

$$= \sum_{r=0}^{n} \frac{\phi(x)}{(x - x_r)} \frac{f(x_r)}{\phi'(x_r)} .$$ **Proved.**

Q. 2. (b) Derive Lagrange's interpolation formula for unequal intervals. (Chhatarpur 2022)

Solution : Let $y = f(x)$ be a function. Suppose $y_0, y_1, \ldots\ldots, y_n$ are $(n + 1)$ values of y corresponding to $(n + 1)$ values $x_0, x_1, \ldots\ldots, x_n$ of the variable x. Then

$$y = f(x) = P_n(x) = A_0(x - x_1)(x - x_2)\ldots\ldots(x - x_n) + A_1(x - x_0)(x - x_2)\ldots\ldots(x - x_n) + A_2(x - x_0)(x - x_1)\ldots\ldots(x - x_n) + \ldots\ldots + A_n(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1}) \quad \ldots(1)$$

Clearly eqn. (1) is n degree polynomial in which (x, y) is replaced by (x_i, y_i); $i = 0, 1, 2, \ldots\ldots n$. We get the values of a coefficient $A_0, A_1, A_2, \ldots\ldots, A_n$ as follows :

$$\left.\begin{aligned} y_0 &= A_0(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n) \\ y_1 &= A_1(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_1 - x_n) \\ y_2 &= A_2(x_2 - x_0)(x_2 - x_1)\ldots\ldots(x_2 - x_n) \\ &\ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ y_n &= A_n(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1}) \end{aligned}\right\} \quad \ldots(2)$$

Eliminating $A_0, A_1, A_2, \ldots\ldots, A_n$ from eqns. (1) and (2), we get

$$y = \frac{(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots\ldots(x_0 - x_n)} y_1 + \ldots\ldots + \frac{(x - x_0)(x - x_1)\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots\ldots(x_n - x_{n-1})} y_n \quad \ldots(3)$$

$$\Rightarrow y = \sum_{i=0}^{n'} \frac{(x - x_0)(x - x_1)(x - x_2)\ldots\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)(x_i - x_2)\ldots\ldots(x_i - x_n)} \frac{y_i}{(x - x_i)} \quad (4)$$

Where $\sum'$ means neglecting the factor $x_i - x_i$ from the denominator

If $\pi = (x-x_0)(x-x_1).....(x-x_{i-1})(x-x_i)(x-x_{i+1}).....(x-x_n)$

Then, $\pi'(x_i) = \left[\frac{d}{dx}\pi(x)\right]_{x=x_i}$

$= (x-x_0)(x-x_1).....(x-x_{i-1})(x-x_{i+1}).....(x-x_n)$

Hence, eqn. (3) can be written as

$$y = P_n(x) = \sum_{i=0}^{n} \frac{\pi(x)}{\pi'(x_i)} \frac{y_i}{(x-x_i)} \quad ...(5)$$

The formulae (4) and (5) are the different form of the formula (3).

This is the required Lagrange's formula.

Q. 3. (a) Solve :

$$(1+\Delta)(1-\nabla) = 1.$$ **(Indore 2022)**

Solution : $(1+\Delta)(1-\nabla)f(x)$

$= (1+\Delta)[(1-\nabla)f(x)]$

$= (1+\Delta)[f(x) - \nabla f(x)]$

$= (1+\Delta)[f(x) - (f(x) - f(x-h))]$

$= (1+\Delta)[f(x) - f(x) + f(x-h)]$

$= (1+\Delta)f(x-h)$

$= Ef(x-h)$

$= f(x)$

$= 1.f(x)$

$\therefore \quad (1+\Delta)(1-\nabla) = 1.$ **Ans.**

Q. 3. (b) Show that :

$$\Delta^2 = E^2 - 2E + 1.$$ **(Indore 2022)**

Solution : $\Delta^2 f(x) = \Delta[\Delta f(x)]$

$= \Delta[f(x+h) - f(x)]$

$= \Delta f(x+h) - \Delta f(x)$

$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$

$= f(x+2h) - f(x+h) - f(x+h) + f(x)$

$= f(x+2h) - 2f(x+h) + f(x)$

$= E^2 f(x) - 2E f(x) + f(x)$

$= (E^2 - 2E + 1)f(x)$

$\therefore \quad \Delta^2 = E^2 - 2E + 1.$ **Proved.**

Q. 4. (a) By using lagrange in iterpolation formula compute the value $f(323{\cdot}5)$ from given a table of values of the function $y = f(x)$

x	y
321·0	2·50651
322·8	2·50893
324·2	2·51081
325·0	2·51188.

(Gwalior 2011; Bhopal 13)

Solution : The lagrange's interpolation formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$x_0 = 321{\cdot}0, f(x_0) = f(321{\cdot}0) = 2{\cdot}50651,$

$x_1 = 322{\cdot}8, \ f(x_1) = f(322{\cdot}8) = 2{\cdot}50893,$

$x_2 = 324{\cdot}2, f(x_2) = f(324{\cdot}2) = 2{\cdot}51081,$

$x_3 = 325{\cdot}0, \ f(x_3) = f(325{\cdot}0) = 2{\cdot}51188$ and $x = 323{\cdot}5$

Putting the values, we have,

$f(325{\cdot}0) = 2{\cdot}50987.$ **Ans.**

Q. 4. (b) Using Lagrange's interpolation formula find the value of $f(x)$ for $x = 7$. **(Indore 2011)**

x	0	2	5	8
$f(x)$	6	10	12	16

Solution : The Lagrang'es interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

Here $x_0 = 0, x_1 = 2, x_2 = 5, x_3 = 8$

$f(x_0) = 6, f(x_1) = 10, f(x_2) = 12, f(x_3) = 16$ and $x = 7$

Putting the values, we have

$$f(7) = \frac{(7-2)(7-5)(7-8)}{(0-2)(0-5)(0-8)} \times 6 + \frac{(7-0)(7-5)(7-8)}{(2-0)(2-5)(2-8)} \times 10$$

$$+\frac{(7-0)(7-2)(7-8)}{(5-0)(5-2)(5-8)} \times 12 + \frac{(7-0)(7-2)(7-5)}{(8-0)(8-2)(8-5)} \times 16$$

$$= \frac{-60}{-80} + \frac{-140}{36} + \frac{-420}{-45} + \frac{1120}{144}$$

$$= \frac{3}{4} - \frac{35}{9} + \frac{84}{9} + \frac{70}{9}$$

$$= \frac{27 - 140 + 336 + 280}{36}$$

$$= \frac{503}{36} = 13{\cdot}97. \qquad \textbf{Ans.}$$

Q. 5. Using the Lagrange's interpolation formula, find the value of $y(10)$ from the following table **(Ujjain 2016; Bhopal 17, 22)**

x	5	6	9	11
y	12	13	14	16

Solution : From Lagrange's interpolation formula, we know that

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)} \times 13$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)} \times 16$$

Taking $x = 10$,

$$y(10) = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)} \times 12 + \frac{(5)(1)(-1)}{(1)(-3)(-5)} \times 13 + \frac{(5)(4)(-1)}{(4)(3)(-2)} \times 14$$

$$+ \frac{(5)(4)(1)}{(6)(5)(2)} \times 16$$

$$= 14{\cdot}666666. \qquad \textbf{Ans.}$$

Q. 6. Apply Lagrange's formula to find $f(5)$, given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8, f(4) = 16, f(7) = 128$ and explain why the result differs from 2^5.

Solution : Given data can be written in tabular form as :

x	1	2	3	4	5	7
$f(x)$	2	4	8	16	?	128
2^x	2	4	8	16	32	128

By Lagrange's interpolation formula, we have

$$f(5) = \frac{(5-2)(5-3)(5-4)(5-7)}{(1-2)(1-3)(1-4)(1-7)} \times 2 + \frac{(5-1)(5-3)(5-4)(5-7)}{(2-1)(2-3)(2-4)(2-7)} \times 4$$

$$+ \frac{(5-1)(5-2)(5-4)(5-7)}{(3-1)(3-2)(3-4)(3-7)} \times 8 + \frac{(5-1)(5-2)(5-3)(5-7)}{(4-1)(4-2)(4-3)(4-7)} \times 16$$

$$+ \frac{(5-1)(5-2)(5-3)(5-4)}{(7-1)(7-2)(7-3)(7-4)} \times 128$$

$$= -\frac{24}{36} + \frac{64}{10} - \frac{192}{8} + \frac{768}{18} + \frac{3072}{360}$$

$$= -0{\cdot}66 + 6{\cdot}4 - 24 + 42{\cdot}67 + 8{\cdot}53 = 32{\cdot}93 \neq 32 = 2^5.$$

Here we have calculated the value of $f(5)$ under the assumption that $f(x)$ can be expressed as polynomial in x but in this problem $f(x)$ is given of the form 2^x.

Hence, the value differs from 2^5. **Ans.**

Q. 7. Evaluate : $\Delta[e^{ax}.\log(bx+c)]$ **(Jabalpur 2011)**

Solution : $\Delta[e^{ax}.\log(bx+c)] = e^{a(x+h)} \log[b(x+h)+c] - e^{ax}\log(bx+c)$

Add and subtract $e^{a(x+h)}.\log(bx+c)$

$$\Delta[e^{ax}\log(bx+c)] = e^{a(x+h)}\log b(x+h) + c$$

$$- e^{a(x+h)}\log(bx+c) + e^{a(x+h)}\log(bx+c) - e^{ax}\log(bx+c)$$

$$= e^{a(x+h)}.\log\frac{b(x+h)+c}{b(x+c)} + e^{ax}(e^{ah}-1)\log(bx+c)$$

$$= e^{ax}\left[e^{ah}\log\frac{b(x+h)+c}{bx+c} + (e^{ah}-1)\log(bx+c)\right]. \qquad \textbf{Ans.}$$

Q. 8. Find the 7th term and also general term of the sequence 2, 9, 28, 65, 126, 127. **(Jabalpur 2016)**

Solution : The forward difference table is given below :

x	$y = f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
x_0	2				
		7			
x_1	9		12		
		19		6	
x_2	28		18		0
		37		6	
x_3	65		24		0
		61		6	
x_4	126		30		
		91			
x_5	227				

If $f_n = f(x_n)$ is a general term then

$$f_n = f_0 + {}^nC_1\Delta f_0 + {}^nC_2\Delta^2 f_0 + \ldots + {}^nC_n\Delta^n f_0$$

$$= 2 + 7\,{}^nC_1 + 12\,{}^nC_2 + 6\,{}^nC_3$$

$$= 2 + 7n + \frac{n\times(n-1)}{2}12 + \frac{n(n-1)(n-2)}{6}.6$$

$$= n^3 + 3n^2 + 3n + 2$$

We get 7th term for x_6. Hence $f_6 = f(x_6)$ is a 7th term which is

$$f_6 = 6^3 + 3\times6^2 + 3\times6 + 2 = 344.$$ **Ans.**

Q. 9. Find a polynomial satisfied by (4, 48), (5, 100), (7, 294), (10, 900), (11, 1210), (13, 2028) by use of Newton's interpolation formula with divided difference. Also, find $f(x)$ for $x = 2, 8, 15$. **(Indore 2017)**

Solution : The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
5	100	$\frac{100-48}{5-4}=52$			
7	294	$\frac{294-100}{7-5}=97$	$\frac{97-52}{7-4}=15$		
10	900	$\frac{900-294}{10-7}=202$	$\frac{202-97}{10-5}=21$	$\frac{21-15}{10-4}=1$	
11	1210	$\frac{1210-900}{11-10}=310$	$\frac{310-202}{11-7}=27$	$\frac{27-21}{11-5}=1$	$\frac{1-1}{11-4}=0$
13	2028	$\frac{2028-1210}{13-11}=409$	$\frac{409-310}{13-10}=33$	$\frac{33-27}{13-7}=1$	$\frac{1-1}{13-5}=0$

Newton's divided difference formula for arguments 4, 5, 7, 10, 11, 13 is

$$f(x) = f(4) + (x-4)\underset{5}{\Delta} f(4) + (x-4)(x-5)\underset{5,7}{\Delta^2} f(4)$$

$$+ (x-4)(x-5)(x-7)\underset{5,7,10}{\Delta^3} f(4) + (x-4)(x-5)(x-7)(x-10)\underset{5,7,10,11}{\Delta^4} f(4)$$

$$+ (x-4)(x-5)(x-7)(x-10)(x-11)\underset{5,7,10,11,13}{\Delta^5} f(4)$$

Putting the value of divided difference, we get

$$f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1) + 0 + 0.$$ **Ans.**

Put $x = 8$, we have

$$f(8) = 48 + 208 + 180 + 12 = 448$$

Put $x = 9$, we have

$$f(9) = 48 + (9-4)(52) + (9-4)(9-5)(15) + (9-4)(9-5)(9-7)(1)$$

$$= 48 + 260 + 300 + 40 = 648$$

and put $x = 15$, we have

$$f(15) = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1)$$

$$= 48 + 572 + 1650 + 880$$

$$= 3150.$$ **Ans.**

Q. 10. Find sin 52° using the following table and also estimate the error by using Newton's forward difference interpolation formula. **(Bhopal 2016; Rewa 16)**

x	45°	50°	55°	60°
$y = \sin x$	0·7071	0·7660	0·8192	0·8660

Solution : The table of finite difference is :

x	$y = f(x) = \sin x$	Δf	$\Delta^2 f$	$\Delta^3 f$
45°	0·7071			
		0·0589		
50°	0·7660		−0·0057	
		0·0532		−0·0007
55°	0·8192		−0·0064	
		0·0468		
60°	0·8660			

$$y = f(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{6}\Delta^3 f_0 + \ldots$$

Where $s = \frac{52-45}{5} = 1{\cdot}4$

$$\sin 52 = f(52) = 0{\cdot}7071 + (1{\cdot}4)(0{\cdot}0589) + \frac{(1{\cdot}4)(0{\cdot}4)}{2}(-0{\cdot}0057)$$

$$+ \frac{(1{\cdot}4)(0{\cdot}4)(-0{\cdot}6)}{6}(-0{\cdot}007)$$

$$= 0{\cdot}7071 + 0{\cdot}08246 - 0{\cdot}001596 + 0{\cdot}0000392$$

$$\sin 52° = 0{\cdot}7880032$$

$$= \frac{s(s-1)(s-2)\ldots(s-n)}{(n+1)!}\Delta^{n+1} f(\xi)$$

$$= \frac{(1{\cdot}4)(0{\cdot}4)(1{\cdot}4-2)}{3!}\Delta^3 f(\xi), \text{ if } n = 2 \text{ then we have}$$

$$= \frac{(1{\cdot}4)(0{\cdot}4)(-0{\cdot}6)}{6}(-0{\cdot}0007) = 0{\cdot}0000392$$

Q. 11. Prove that $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$. **(Indore 2011, 13, 14, 16; Gwalior 14, 16; Bhopal 16; Ujjain 22; Chhindwara 22)**

Solution : We have

$$\Delta \log f(x) = \log f(x+h) - \log f(x)$$

$$= \log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[\frac{Ef(x)}{f(x)}\right], \qquad (\because Ef(x) = f(x+h))$$

$$= \log\left[\frac{(1+\Delta)f(x)}{f(x)}\right], \qquad (\because E = 1+\Delta)$$

$$= \log\left[\frac{f(x)+\Delta f(x)}{f(x)}\right] = \log\left[1+\frac{\Delta f(x)}{f(x)}\right].$$ **Proved.**

Q. 12. Find the expected age from the following table to 16 years :

Age	**10**	**15**	**20**	**25**	**30**	**35**
Expected age (in years)	**35·4**	**32·3**	**29·2**	**26·0**	**23·2**	**20·4**

Solution : Here $x = 16, x_0 = 10$.

The table of finite difference is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
10	35·4						
		−3·1					
15	32·3		0				
		−3·1		−0·1			
20	29·2		−0·1		0·6		
		−3·2		0·5		−1·5	
25	26·0		0·4		−0·9		
		−2·8		−0·4			
30	23·2		0				
		−2·8					
35	20·4						

$$u = \frac{x - x_0}{h} = \frac{16-10}{5} = 1{\cdot}2$$

$$y_{1\cdot2} = E^{1\cdot2} y_0 = (1+\Delta)^{1\cdot2} y_0$$

$$= y_0 + \frac{1\cdot2}{1}\Delta y_0 + \frac{(1\cdot2)(1\cdot2-1)}{2!}\Delta^2 y_0 + \frac{(1\cdot2)(1\cdot2-1)(1\cdot2-2)}{3!}\Delta^3 y_0$$

$$+ \frac{(1\cdot2)(1\cdot2-1)(1\cdot2-2)(1\cdot2-3)}{4!}\Delta^4 y_0$$

$$+ \frac{(1\cdot2)(1\cdot2-1)(1\cdot2-2)(1\cdot2-3)(1\cdot2-4)}{5!}\Delta^5 y_0$$

$$= 35{\cdot}4 + (1{\cdot}2)(-3{\cdot}1) + \frac{(1{\cdot}2)(0{\cdot}2)}{2}0 + \frac{(1{\cdot}2)(0{\cdot}2)(-0{\cdot}8)}{6}(-0{\cdot}1)$$

$$+ \frac{(1{\cdot}2)(0{\cdot}2)(-0{\cdot}8)(-1{\cdot}8)^2}{24}(0{\cdot}6) + \frac{(1{\cdot}2)(0{\cdot}2)(-0{\cdot}8)(-1{\cdot}8)(-2{\cdot}8)}{120}(-1{\cdot}5)$$

$$= 35{\cdot}4 - 3{\cdot}72 + 0 + 0{\cdot}0032 + 0{\cdot}00864 + 0{\cdot}0012$$

$$= 31{\cdot}693.$$ **Ans.**

Q. 13. Find the values of $f(2)$, $f(8)$ and $f(15)$ from the following table by means of Newton's divided difference formula.

(Indore 2011; Bhopal 14; Jabalpur 16; Rewa16; Ujjain 22)

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Solution : The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		$\frac{100-48}{5-4} = 52$			
5	100				
		$\frac{294-100}{7-5} = 97$	$\frac{97-52}{7-4} = 15$		
7	294				
		$\frac{900-294}{10-7} = 202$	$\frac{202-97}{10-5} = 21$	$\frac{21-15}{10-4} = 1$	
10	900				
		$\frac{1210-900}{11-10} = 310$	$\frac{310-202}{11-7} = 27$	$\frac{27-21}{11-5} = 1$	$\frac{1-1}{11-4} = 0$
11	1210				
		$\frac{2028-1210}{13-11} = 409$	$\frac{409-310}{13-10} = 33$	$\frac{33-27}{13-7} = 1$	$\frac{1-1}{13-5} = 0$
13	2028				

Newton's divided difference formula for arguments 4, 5, 7, 10, 11, 13 is

$$f(x) = f(4) + (x-4)\underset{5}{\Delta} f(4) + (x-4)(x-5)\underset{5,7}{\Delta^2} f(4)$$

$$+ (x-4)(x-5)(x-7)\underset{5,7,10}{\Delta^3} f(4) + (x-4)(x-5)(x-7)(x-10)\underset{5,7,10,11}{\Delta^4} f(4)$$

$$+ (x-4)(x-5)(x-7)(x-10)(x-11)\underset{5,7,10,11,13}{\Delta^5} f(4)$$

Putting the value of divided difference, we get

$$f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1) + 0 + 0.$$ **Ans.**

Put $x = 8$, we have

$$f(8) = 48 + 208 + 180 + 12 = 448$$

Put $x = 9$, we have

$$f(9) = 48 + (9-4)(52) + (9-4)(9-5)(15) + (9-4)(9-5)(9-7)(1)$$

$$= 48 + 260 + 300 + 40 = 648$$

and put $x = 15$, we have

$$f(15) = 48 + (15-4)(52) + (15-4)(15-5)(15) + (15-4)(15-5)(15-7)(1)$$

$$= 48 + 572 + 1650 + 880 = 3150.$$ **Ans.**

Q. 14. Given :

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Find $f(7{\cdot}5)$.

Solution : Since the value 7·5 is much nearer to last tabulated value *i.e.*, $x = 8$, so we can use Newton's backward interpolation formula

Here $x = 7{\cdot}5$, $a + nh = 8$, $h = 1$

$$\therefore \quad u = \frac{x-(a+nh)}{h} = \frac{7{\cdot}5-8}{1} = -0{\cdot}5$$

The table of backward finite difference is :

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		0
		91		6	
6	216		36		0
		127		6	
7	343		42		
		169			
8	512				

Applying Newton's backward difference formula, we have

$$f(a+nh+uh) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{\lfloor 2}\nabla^2 f(a+nh)$$

$$+\frac{u(u+1)(u+2)}{\lfloor 3}\nabla^3 f(a+nh)$$

$\therefore$ $\nabla^3 f(x)$ is constant, then

$$f(7{\cdot}5) = f(8) + (-0{\cdot}5)\nabla f(8) + \frac{(-0{\cdot}5)(-0{\cdot}5+1)}{2}\nabla^2 f(8)$$

$$+\frac{(-0{\cdot}5)(-0{\cdot}5+1)(-0{\cdot}5+2)}{6}\nabla^2 f(8)$$

$$= 512 + (-0{\cdot}5)\times 169 + \frac{(-0{\cdot}5)(0{\cdot}5)}{2}\times 42 + \frac{(-0{\cdot}5)(0{\cdot}5)(1{\cdot}5)}{6}\times 6$$

$$= 421{\cdot}875.$$

Ans.

Q. 15. Evaluate $f(9)$ by using Newton's divided difference formula.

(Rewa 2011, 14, 17; Indore 14, 16, 22; Jabalpur 17)

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Solution : Divided difference table is :

x	$f(x)$	divided difference			
		$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150	$\frac{392-150}{7-5} = 121$			
			$\frac{265-121}{11-5} = 24$		
7	392	$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
			$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
11	1452	$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
			$\frac{709-457}{17-11} = 42$		
13	2366	$\frac{5202-2366}{17-13} = 709$			
17	5202				

By Newton's divided difference formula, we have

$$f(x) = f(5) + (x-5)\underset{7}{\Delta} f(5) + (x-5)(x-7)\underset{7,11}{\Delta^2} f(5)$$

$$+(x-5)(x-7)(x-11)\underset{7,11,13}{\Delta^3} f(5) + (x-5)(x-7)(x-11)(x-13)\Delta^4 f(5)$$

$$= 150 + (x-5)121 + (x-5)(x-7)24 + (x-5)(x-7)(x-11)$$

$$+(x-5)(x-7)(x-11)(x-13)\times 0$$

Hence

$$f(9) = 150 + 4\times 121 + (9-5)(9-7)24 + (9-5)(9-7)(9-11)\times 1$$

$$= 150 + 484 + 8\times 24 + (-16) = 810.$$

Ans.

Q. 16. Construct a table of divided difference for the given following value and find the value of $f(7)$. **(Bhopal 2011)**

x	0·5	1·5	3·0	5·0	6·5	8·0
$f(x)$	1·625	5·875	31·0	131·0	282·125	521·0

Solution : The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0·5	1·625	$\frac{5·875-1·625}{1·5-0·5}=4·25$			
1·5	5·875		$\frac{16·75-4·25}{3·0-0·5}=5$	$\frac{9·5-5}{5·0-0·5}=1$	$\frac{1-1}{6·5-0·5}=0$
3·0	31·0	$\frac{31·0-5·875}{3·0-1·5}=16·75$	$\frac{50·0-16·75}{5·0-1·5}=9·5$	$\frac{14·5-9·5}{6·5-1·5}=1$	
5·0	131·0	$\frac{131-31}{5·0-3·0}=50$	$\frac{100·75-50}{6·5-3·0}=14·5$		
6·5	282·125	$\frac{282·125-131}{6·5-5·0}=100·75$		$\frac{19·5-14·5}{8-3}=1$	$\frac{1-1}{8-1·5}=0$
8·0	521·0	$\frac{521·0-282·125}{8·0-6·5}=159·25$	$\frac{159·25-100·75}{8·0-5·0}=19·5$		

By Newton's divided difference formula

$$f(7)=f(0·5)+(7-0·5)\underset{1·5}{\Delta}f(0·5)+(7-0·5)(7-1·5)\underset{1·5,3}{\Delta^2}f(0·5)$$
$$+(7-0·5)(7-1·5)(7-3)\underset{1·5,3,5}{\Delta^3}f(0·5)+(7-0·5)(7-1·5)$$
$$(7-3)(7-5)\Delta^4 f(0·5)+0+0$$
$$=1·625+(6·5)\times4·25+6·5\times5·5\times5+6·5\times5·5\times4\times1$$
$$+6·5\times5·5\times4\times2\times0+0$$
$$=351.$$ **Ans.**

Q. 17. (a) The population of a country in the decennial censuces we as under. Estimate the population for the year 1925. **(Jabalpur 2011, 14)**

Year x	**1891**	**1901**	**1911**	**1921**	**1931**
Population y (In Lakh)	**46**	**66**	**81**	**93**	**101**

Solution : The divided difference table is :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
1901	66	20	−5		
1911	81	15	−3	2	−3
1921	93	12	−4	−1	
1931	101	08			

By the Newton's forward interpolation formula

$$f(a+uh)=f(x)=f(a)+u.\Delta f(a)+\frac{u(u-1)}{2!}\Delta^2 f(a)$$
$$+\frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)+\ldots\ldots$$

Here $a=1891, h=10, x=1925$ and $u=\frac{x-a}{h}=\frac{1925-1891}{10}$

Hence $f(1925)=46+3·4\times20+\frac{(3·4)(3·4-1)}{2!}(-5)+\frac{3·4\times(3·4-1)(3·4-2)}{3!}\times2$
$$+\frac{(3·4)(3·4-1)(3·4-2)(3·4-3)}{4!}\times(-3)$$
$$=46+68-20·4+3·808-0·5712$$
$$=96·8368 \text{ Lakh.}$$ **Ans.**

Q. 17. (b) Estimate the sale for 1966 using the following table : **(Jabalpur 2022)**

Year	**1931**	**1941**	**1951**	**1961**	**1971**	**1981**
Sale (in thousands)	**12**	**15**	**20**	**27**	**39**	**52**

Solution : By Newton's forward interpolation formula :

Year (x)	**Sale (y)**	**Differences**				
		$\Delta f(a)$	$\Delta^2 f(a)$	$\Delta^3 f(a)$	$\Delta^4 f(a)$	$\Delta^5 f(a)$
1931	$12=f(a)$					
		3				
1941	15		2			
		5		0		
1951	20		2		3	
		7		3		−10
1961	27		5		−7	
		12		−4		
1971	39		1			
		13				
1981	52					

$$f(x)=f(a+uh)=f(a)+u\Delta f(a)+\frac{u(u-1)}{\lfloor 2}\Delta^2 f(a)+\ldots\ldots \quad \ldots(1)$$

Here $a=1931, h=10, x=1966$

$\therefore$ $$u=\frac{x-a}{h}=\frac{1966-1931}{10}=3·5$$

By eqn. (1)

$$f(1966)=12+(3·5)\times3+\frac{(3·5)(2·5)}{\lfloor 2}\times2+\frac{(3·5)(2·5)(1·5)}{\lfloor 3}\times0$$
$$+\frac{(3·5)(2·5)(1·5)(0·5)}{\lfloor 4}\times3+\frac{(3·5)(2·5)(1·5)(0·5)(-0·5)}{\lfloor 5}\times(-10)$$

$= 12 + 10{\cdot}5 + 8{\cdot}75 + 0{\cdot}8203 + 0{\cdot}2734$

$= 32{\cdot}3437$

Hence the sale for year 1966 is 32·3437 thousands. **Ans.**

Q. 18. Find the missing term : **(Indore 2017)**

x	1	2	3	4	5
$f(x)$	7	—	13	21	37

Solution : Since only four values of $f(x)$ are given one term if $f(x)$ is missing when $x = 2$, so polynomial will be degree of 3.

Hence $\Delta^3 f(x) =$ Constant and $\Delta^{3+k} f(x) = 0, \forall\, k \geq 1$ in the form of

$$\Delta^4 f(x) = 0$$

Or $$(E-1)^4 f(x) = 0$$

Or $$(E^4 - 4E^3 + 6E^2 - 4E + 1) f(x) = 0$$

Or $$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

Put $x = 1$, we get

$$f(5) - 4f(4) + 6f(3) - 4f(2) + f(1) = 0$$

Or $$37 - 4\times 21 + 6\times 13 - 4f(2) + 7 = 0$$

Or $$38 - 4f(2) = 0$$

$\therefore$ $$f(2) = 9{\cdot}5$$

Here we assume that $\Delta^n f(x) = 0$ but for $2^x + 5, \Delta^n f(x)$ not be zero.

Hence the missing term is 9·5. **Ans.**

Q. 19. Prove that :

$$\Delta \equiv \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}.$$ **(Ujjain 2017)**

Solution : $\dfrac{1}{2}\delta^2 + \delta\sqrt{1+\dfrac{\delta^2}{4}} = \dfrac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{1 + \dfrac{(E^{1/2} - E^{-1/2})^2}{4}}$

$$= \frac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= \frac{E + E^{-1} - 2}{2} + (E^{1/2} - E^{-1/2})\sqrt{\frac{(E^{1/2} + E^{-1/2})^2}{4}}$$

$$= \frac{E + E^{-1} - 2}{2} + \frac{(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})}{2}$$

$$= \frac{E + E^{-1}}{2} - 1 + \frac{E - E^{-1}}{2}$$

$$= E - 1 = \Delta$$ **Proved.**

Q. 20. (a) Find $f'(5)$ by using the following table :

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

(Jabalpur 2016)

Solution : Since the value of x are not equal. Hence we will use the Newton's divided difference interpolation formula, which is following :

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots\ldots \ldots(1)$$

Differentiating (1) w.r. to x

$$f'(x) = 0 + f(x_0, x_1) + \{(x - x_0) + (x - x_1)\} f(x_0, x_1, x_2) + \{(x - x_0)(x - x_1)$$
$$+ (x - x_1)(x - x_2) + (x - x_0)(x - x_2)\} f(x_0, x_1, x_2, x_3) + \ldots\ldots \ldots(2)$$

Divided Difference Table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
0	4					
		$\dfrac{26-4}{2-0} = 11$				
2	26		$\dfrac{32-11}{3-0} = 7$			
		$\dfrac{58-26}{3-2} = 32$		$\dfrac{11-7}{4-0} = 1$		
3	58		$\dfrac{54-32}{4-2} = 11$		0	
		$\dfrac{112-58}{4-3} = 54$		$\dfrac{16-11}{7-2} = 1$		0
4	112		$\dfrac{118-54}{7-3} = 16$			
		$\dfrac{466-112}{7-4} = 118$		$\dfrac{22-16}{9-3} = 1$	0	
7	466		$\dfrac{228-118}{9-4} = 22$			
		$\dfrac{922-466}{9-7} = 228$				
9	922					

Hence taking $x = 5, x_0 = 0, x_1 = 2, x_2 = 3, x_3 = 4, x_4 = 7, x_5 = 9,$ by eqn. (2)

$f'(5) = 0 + 11 + 8\times 7 + (15 + 6 + 10)\times 1 = 11 + 56 + 31 = 98.$ **Ans.**

Q. 20. (b) Find $f'(4)$ by the following table : **(Jabalpur 2016)**

x	1	2	4	8	10
$f(x)$	0	1	5	21	27

Solution : Since the interval of x is unequal. So by the Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \ldots\ldots$$

$$f'(x) = 0 + f(x_0, x_1) + \{(x - x_1) + (x - x_0)\} f(x_0, x_1, x_2) + \{(x - x_1)(x - x_2) + (x - x_2)(x - x_0) + (x - x_0)(x - x_1)\} f(x_0, x_1, x_2, x_3) + \ldots\ldots \quad \ldots(1)$$

$x_0 = 1, x_1 = 2, x_3 = x = 4, x_4 = 8, x_5 = 10.$

Putting value of the divided difference from the table in eqn. (1), we get

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		1			
2	1		$\frac{1}{3}$		
		2		0	
4	5		$\frac{2}{6} = \frac{1}{3}$		0
		4		0	
8	21		0		
		4			
10	27				

$$f'(4) = 1 + \{(4-1) + (4-2)\}\frac{1}{3} + 0 + 0$$

$$= 1 + \frac{5}{3} = \frac{8}{3} = 2{\cdot}67.$$ **Ans.**

Q. 21. Find the value of $f'(6)$ and the highest value of $f(x)$ from the following table :

x	0	2	3	4	7	9
$y = f(x)$	4	26	58	112	466	922

(Bhopal 2017)

Solution : By the Newton's divide difference formula divide difference table :

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	4				
		11			
2	26		7		
		32		1	
3	58		11		0
		54		1	
4	112		16		0
		118		1	
7	466		22		
		228			
9	922				

By Newton's formula, we have

$$y = f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1)\ f(x_0, x_1, x_2) + \ldots.$$

$$= 4 + (x - 0)11 + (x - 0)(x - 2)7 + (x - 0)(x - 2)(x - 3)$$

$$= x^3 + 2x^2 + 3x + 4$$

$$\therefore \quad f'(x) = 3x^2 + 4x + 3$$

$$\therefore \quad f'(c) = 3c^2 + 4c + 3$$

$$f'(6) = 3(6)^2 + 4(6) + 3 = 135$$

If $f'(x) = 0$ then $f(x)$ is highest

$$3x^2 + 4x + 3 = 0$$

But the root of equation is imaginary

$\therefore$ The value of $f(x)$ is not in this interval

$\therefore$ $f(x)$ is a highest value curve. **Ans.**

Q. 22. Write the properties of Cote's Number. Also prove them.
(Rewa 2016; Jabalpur 16; Gwalior 16, 17)

(i) Symmetric property of Cote's numbers

i.e., to prove that $C_k^n = C_{n-k}^n$.

(ii) To show that $\sum_{k=0}^{n} C_k^n = 1$.

Or

Prove that the sum of Cote's number is unity.

Solution : (i) We have

$$C_k^n = \frac{1}{n}\int_0^n L_k\, du = \frac{1}{n}\int_0^n \frac{u(u-1)\ldots.\{u-(k-1)\}\{u-(k+1)\}\ldots..(u-n)}{k(k-1)\ldots(1)(-1)(-2)\ldots(k-n)}\, du$$

$$= \frac{1}{n}\int_0^n \frac{u(u-1)\ldots.\{u-(k-1)\}\{u-(k+1)\}\ldots..(u-n)}{k!(-1)^{n-k}(n-k)!}\, du. \quad \ldots(1)$$

Now putting $u - n = -t \Rightarrow u = n - t$,

$\therefore$ $du = -dt$ and replacing k by $n - k$ in eqn. (1), we have

$$C_{n-k}^n = \frac{1}{n}\int_n^0 \frac{(n-t)(n-t-1)\ldots(-t+k+1)(-t+k-1)\ldots(-t)}{(n-k)!(-1)^k(k!)}(-dt)$$

$$= \frac{1}{n}\int_0^n \frac{(-1)^n t(t-1)\ldots\{t-(k-1)\}\{t-(k+1)\}\ldots(t-n)}{(-1)^k(k)!(n-k)!}\, dt$$

$$= \frac{1}{n}\int_0^n \frac{(-1)^{n-k} t(t-1)\ldots\{t-(k-1)\}\{t-(k+1)\}\ldots(t-n)}{(k)!(n-k)!}\, dt$$

$$= \frac{1}{n}\int_0^n \frac{t(t-1)\ldots\{t-(k-1)\}\{t-(k+1)\}\ldots(t-n)}{(k)!.(-1)^{n-k}(n-k)}\, dt,$$

[on multiplying Nr and Dr by $(-1)^{n-k}$ and noting $(-1)^{2(n-k)} = +1$]

$= C^n_{n-k}$. $\left[\text{in veiw of eqn.(1) and } \int_a^b f(u)du = \int_a^b f(t)dt\right]$

Proved.

(ii) We have

$$\sum_{k=0}^{n} C_k^n = C_0^n + C_1^n + \ldots + C_n^n = \frac{1}{n}\left[\int_0^n (L_0 + L_1 + \ldots L_n)\,du\right], \quad \ldots(1)$$

But

$$f(x) = P(x) = \sum_{k=0}^{n} L_x(x) y_k \quad \ldots(2)$$

Where y_k is the value of $P(x)$ at $x_k = x_0 + kh$.

Let $P(x) = (x-\alpha)^i$, then we have

$$y_k = (x_k - \alpha)^i.$$

Putting these values of $P(x)$ and y_k in eqn. (2), we have

$$(x-\alpha)^i = \sum_{k=0}^{n} (x_k - \alpha)^i L_k(x), \text{for } i = 1,2,\ldots n.$$

Putting $i = 0$ in above equation, we get

$$1 = \sum_{k=0}^{n} L_k \Rightarrow 1 = (L_0 + L_1 + \ldots + L_n) \quad \ldots(3)$$

Using eqn. (3) in eqn. (1), we have

$$\sum_{k=0}^{n} C_k^n = \frac{1}{n}\int_0^n 1\,du = \frac{1}{n}[n] = 1$$

$\therefore \quad \sum_{k=0}^{n} C_k^n = 1.$ Proved.

Q. 23. Derive Simpson $\frac{3}{8}$ rule and estimate the error. (Ujjain 2017)

Solution : Putting $n = 3$ in quadrature formula

$$\int_{x_0}^{x_3} y(dx) = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \frac{9}{4}\Delta^2 y_0 + \frac{3}{8}\Delta^3 y_0\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_3}^{x_6} y\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6)$$

....

Continuing the process

$$\int_{x_{n-3}}^{x_n} y\,dx = \frac{3h}{8}(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)$$

Adding all integrals

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})] \quad \ldots(1)$$

Error evaluation :

Let $I = \int_{x_0}^{x_0+3h} f(x)\,dx$ and $\phi(x) = \int f(x)\,dx$

$\therefore \quad I_E = \int_{x_0}^{x_0+3h} f(x)\,dx = \phi(x_0 + 3h) - \phi(x_0)$

$$= 3hf(x_0) + \frac{9}{2}h^2 f'(x_0) + \frac{9}{2}h^3 f''(x_0) + \frac{27}{8}h^4 f'''(x_0) + \frac{81}{40}h^5 f^{iv}(x_0) + \ldots.$$

and $I_S = \frac{3h}{8}[f(x_0) + 3f(x_0 + h) + 3f(x_0 + 2h) + f(x_0 + 3h)]$

$$= 3hf(x_0) + \frac{9}{2}h^2 f'(x_0) + \frac{9}{2}h^3 f''(x_0) + \frac{27}{8}h^4 f'''(x_0) + \frac{33}{16}h^5 f^{iv}(x_0) + \ldots.$$

$\therefore \quad E_S = I_E - I_S = \left(\frac{81}{40} - \frac{33}{16}\right)hf^{iv}(x_0) + \ldots\ldots$

$= -\frac{3}{80}h^5 f^{iv}(x_0)$, (fifth and higher differences are neglect)

This error is for interval $(x_0, x_0 + 3h)$. Total error will be the sum of the error for the intervals like $(x_0 + 3h, x_0 + 6h)$, $\ldots\ldots(x_n - 3h, x_n)$.

$$E_S = -\frac{3}{\delta_0}h^5[f^{iv}(x_0) + f^{iv}(x_3) + \ldots\ldots + f^{iv}(x_{n-3})] \quad \ldots(2)$$

$= -\frac{3}{\delta_0}h^5\frac{n}{3}f^{iv}(\xi)$, where $\xi \in (x_0, x_n)$ and the maximum value of the terms in the bracket of eqn. (2) is $f(\xi)$

Hence $E_S = -\frac{3}{\delta_0}h^4(b-a)f^{iv}(\xi), \xi \in (a,b)$. **Ans.**

Q. 24. Find the value of π from $\int_0^1 \frac{dx}{1+x^2}$ taking eight equal intervals correct to four decimal places. (Gwalior 2012; Indore 13)

Solution : Here $y = \frac{1}{1+x^2}$ and $h = \frac{1-0}{8} = 0{\cdot}125$

Now we find the value of $y = \frac{1}{1+x^2}$ for each point of sub-division and the corresponding computation table is as given below :

x	0	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1·0
$y = \frac{1}{1+x^2}$	1	0·9846	0·9412	0·8767	0·8	0·7191	0·64	0·5664	0·5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Simpson's one-third rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$= \frac{0\cdot125}{3}[(1+0\cdot5)+4(0\cdot9846+0\cdot8767+0\cdot7191+0\cdot5664)$$

$$+2(0\cdot9412+0\cdot800+0\cdot6400)]$$

$$= 0\cdot0417[1\cdot5+4\times3\cdot1468+2\times2\cdot3912]$$

$$= 0\cdot0417[1\cdot5+12\cdot5872+4\cdot7824]$$

$$= 0\cdot0417\times18\cdot8696 = 0\cdot7824 \quad ...(1)$$

Again $\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}(1)-\tan^{-1}0 = \frac{\pi}{4}-0 = \frac{\pi}{4}$ ...(2)

∴ From eqns. (1) and (2), we have

$$\frac{\pi}{4} = 0\cdot7869 \text{ or } \pi = 3\cdot1474.$$ Ans.

Q. 25. Evaluate $\int_0^4 e^x dx$, by Simpson's rule, using the data $e = 2\cdot72$, $e^2 = 7\cdot39$, $e^3 = 20\cdot09, e^4 = 54\cdot60$. (Bhopal 2014)

Solution : Here the range of integration (0, 4) is divided by four equal parts.
Here $h = 1$.

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 e^x dx = \frac{1}{3}h[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

$$= \frac{1}{3}\times1[1+54\cdot60+4(2\cdot72+20\cdot09)+2(7\cdot39)]$$

$$= \frac{1}{3}[161\cdot62] = 53\cdot873$$

The actual value of the integral is

$$\int_0^4 e^x dx = e^4 - e^0 = 54\cdot60 - 1\cdot00 = 53\cdot60.$$ Ans.

Q. 26. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ taking $h = 0\cdot125$ by using trapezoidal rule.

Solution : Here $y = \frac{1}{1+x^2}$ and $h = 0\cdot125$

x	0	0·125	0·25	0·375	0·5	0·625	0·75	0·875	1·0
$y = \frac{1}{1+x^2}$	1	0·9846	0·9412	0·8767	0·8	0·7191	0·64	0·5664	0·5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

By trapezoidal formula, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$

$$= \frac{0\cdot125}{2}[(1+0\cdot5)+2(0\cdot9846+0\cdot9412+0\cdot8767+0\cdot8+0\cdot7191$$

$$+0\cdot64+0\cdot5664)]$$

$$= 0\cdot0625[1\cdot5+11\cdot056] = 0\cdot78475.$$ Ans.

Q. 27. Find the value of $\int_0^1 \frac{1}{1+x^2}dx$

(i) By using trapezoidal rule.

(Gwalior 2011, 13; Jabalpur 11, 16; Ujjain 17; Indore 17)

(ii) By Simpson's $\frac{1}{3}$ rule.

(Bhopal 2013; Jabalpur 14, 16; Rewa 14, 16; Ujjain 14; Chhindwara 22)

(iii) Simpson's $\frac{3}{8}$ rule and find the actual value.

(Jabalpur 2022; Chhindwara 22)

Solution : $a = 0, b = 1, n = 6, \therefore h = \frac{1-0}{6} = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1	0·97297	0·9	0·8	0·69230	0·59016	0·5

(i) By trapezoidal rule :

$$\int_a^b f(x)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+.....+y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{1}{12}[(1+0\cdot5)+2(0\cdot97297+0\cdot9+0\cdot8+0\cdot69230+0\cdot59016)]$$

$$= 0\cdot78424.$$ Ans.

(ii) Simpson's $\frac{1}{3}$ rule :

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+y_5+.....)+2(y_2+y_4+.....)]$$

$$\Rightarrow \int_0^1 \frac{1}{1+x^2}dx = \frac{1}{18}[(1+0\cdot5)+4(0\cdot97297+0\cdot8+0\cdot59016)+2(0\cdot9+0\cdot69230)]$$

$$= 0\cdot78539.$$ Ans.

(iii) Simpson's $\frac{3}{8}$ rule :

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4+y_5+.....)+2(y_3+y_6+.....)]$$

$$= \frac{3}{8}\cdot\frac{1}{6}[(1+0\cdot5)+3(0\cdot97297+0\cdot9+0\cdot69230+0\cdot59016)+2(0\cdot8)]$$

$$= 0\cdot7853593$$

Actual value $= \int_0^1 \frac{1}{1+x^2}dx = (\tan^{-1}x)_0^1 = \frac{\pi}{4}$

$$= \frac{11}{14} = 0\cdot785714.$$ **Ans.**

Q. 28. Evaluate $\int_0^1 \frac{dx}{1+x}$, $h = 0\cdot1$. Hence find the value of log2.

(Indore 2014; Bhopal 16)

Solution : Dividing the whole range (0, 1) into equal 10 parts. Using Simpson's $\frac{1}{3}$ rule, we get

$$x_0 = 0,\ x_n = n, n = 10$$

$$x_n = x_0 + nh \text{ from } h = 0\cdot1$$

So, by Simpson $\frac{1}{3}$ rule $\int_0^1 \frac{dx}{1+x} = \int_{x_0}^{x_0+10h} y dx$

$$= \frac{h}{3}[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)]$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{1}{1\cdot1}+\frac{1}{1\cdot3}+\frac{1}{1\cdot5}+\frac{1}{1\cdot7}+\frac{1}{1\cdot9}\right)+2\left(\frac{1}{1\cdot2}+\frac{1}{1\cdot4}+\frac{1}{1\cdot6}+\frac{1}{1\cdot8}\right)\right]$$

$$= \frac{1}{30}[1\cdot5+4(3\cdot45955)+2(2\cdot72818)] = \frac{1}{30}[20\cdot79456] = 0\cdot69315$$

$$\int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2$$

$$\log 2 = \int_0^1 \frac{dx}{1+x} = 0\cdot69315.$$ **Ans.**

Q. 29. Find the value of $\int_{-3}^{3} x^4 dx$ by Trapezoidal rule. **(Indore 2016)**

***Or*, Calculate $\int_{-3}^{3} x^4 dx$ by using Simpson's $\frac{1}{3}$ rule and Simpson's $\frac{3}{8}$ rule.**

(Ujjain 2016)

Solution : Here $f(x) = x^4$, length of interval $= 3-(-3) = 6$

$$h = \frac{6}{6} = 1$$

x	−3	−2	−1	0	1	2	3
y	81	16	1	0	1	16	81

$$\int_{-3}^{3} f(x)dx = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(81+81)+2(16+1+0+1+16)]$$

$$= 115.$$ **Ans.**

Simpson's $\frac{1}{3}$ rule

$$\int_{-3}^{3} x^4 dx = \frac{h}{3}[(y_0+y_6)+2(y_2+y_4)+3(y_1+y_3+y_5)]$$

$$= \frac{1}{3}[(81+81)+2(1+1)+4(16+0+16)]$$

$$= \frac{1}{3}[(162)+2\times2+4(32)] = \frac{1}{3}[294] = 98.$$ **Ans.**

Simpson's $\frac{3}{8}$ rule

$$\int_{-3}^{3} x^4 dx = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}[(81+81)+3(16+1+1+16)+2.0]$$

$$= \frac{3}{8}[162+102] = \frac{3}{8}\times264 = 99.$$ **Ans.**

Q. 30. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}$ rule.

(Indore 2015)

Solution : Given that $f(x) = \frac{1}{1+x^2}, a = 0, b = 6.$

Since we have to use Simpson's $\frac{1}{3}$ rules, we would require to divide the range (0, 6) in such a manner that it must be divisible by 2 and 3 both.

Therefore, we divide the range of integration into 6 equal parts by taking

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1.$$

Now the values of the given function $y = \frac{1}{1+x^2}$ computed for each point of subdivision are as given below :

x	$y = f(x)$
$x_0 = 0$	$y_0 = 1\cdot00$
$x_0 + h = x_1 = 1$	$y_1 = 0\cdot500$
$x_0 + 2h = x_2 = 2$	$y_2 = 0\cdot200$
$x_0 + 3h = x_3 = 3$	$y_3 = 0\cdot100$
$x_0 + 4h = x_4 = 4$	$y_4 = 0\cdot058824$
$x_0 + 5h = x_5 = 5$	$y_6 = 0\cdot038462$
$x_0 + 6h = x_6 = 6$	$y_6 = 0\cdot027027$

(i) By Trapezoidal rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \int_{x_0}^{x_0+6h} f(x)dx = \int_{x_0}^{x_0+6h} y\,dx$$

$$= \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(1{\cdot}00+0{\cdot}027027)+2(0{\cdot}500+0{\cdot}200+0{\cdot}100)$$

$$+(0{\cdot}058824+0{\cdot}038462)]$$

$$= 1{\cdot}4107995.$$

(ii) By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \int_{x_0}^{x_0+6h} y\,dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{3}[(1{\cdot}00+0{\cdot}027027)+4(0{\cdot}500+0{\cdot}100+0{\cdot}038462)$$

$$+2(0{\cdot}200+0{\cdot}058824)]$$

$$= 1{\cdot}3661743.$$ **Ans.**

Q. 31. Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Gaussian quadrature formula for three ordinates. **(Gwalior 2014)**

Given : $u_1 = -0{\cdot}77459667, u_2 = 0, u_3 = -u_1$

$\omega_1 = 0{\cdot}55555556, \omega_2 = 0{\cdot}88888889, \omega_3 = \omega_1$.

Solution : $I = \int_a^b f(x)dx = \int_0^1 \frac{1}{1+x^2}dx$

Let $x = \frac{b+a}{2} + \left(\frac{b-a}{2}\right)u = \frac{1}{2} + \frac{1}{2}u, dx = \frac{1}{2}du$

$\therefore$ $$I = \frac{1}{2}\int_{-1}^1 \frac{1}{1+\frac{1}{4}(1+u)^2}du$$

$$= \frac{1}{2}\int_{-1}^1 \phi(u)du, \text{ where } \phi(u) = \frac{1}{1+\frac{1}{4}(1+u)^2}$$

$$= \frac{1}{2}\sum_{i=1}^n \omega_i\phi(u_i), n=3$$

$$= \frac{1}{2}[\omega_1\phi(u_1)+\omega_2\phi(u_2)+\omega_3\phi(u_3)]$$

$$= \frac{1}{2}\left[0{\cdot}55555556\times\frac{1}{1+\frac{1}{4}(1-0{\cdot}77459667)^2}+0{\cdot}88888889\right.$$

$$\left.\times\frac{1}{1+\frac{1}{4}(1+0)^2}+0{\cdot}55555556\times\frac{1}{1+\frac{1}{4}(1+0{\cdot}77459667)^2}\right]$$

$$= \frac{1}{2}[0{\cdot}55555556\times0{\cdot}98746+0{\cdot}88888889\times0{\cdot}8$$

$$+0{\cdot}55555556\times0{\cdot}55950]$$

$$= \frac{1}{2}\times1{\cdot}57053 = 0{\cdot}78527.$$ **Ans.**

Q. 32. Evaluate the integral

$$\int_{-1}^1 \frac{1}{1+x^2}dx$$

using Gaussian quadrature formula for three ordinates. **(Indore 2017)**

Solution : Let $I = \int_{-1}^1 \frac{1}{1+x^2}dx = \sum_{i=1}^3 A_i f(t_i)$

Legendre polynomial of degree 3 is

$$P_3(t) = \frac{1}{2}(5t^3-3t)$$

Zeros of this polynomial are obtained from $P_3(t)=0$ *i.e.*,

$$\frac{1}{2}(5t^3-3t)=0$$

or $$t(5t^2-3)=0$$

$\therefore$ Zeros of $P_3(t)$ are $0, \pm\sqrt{\left(\frac{3}{5}\right)}$

Let $$t_1 = -\sqrt{\frac{3}{5}} \approx -0{\cdot}774597$$

$$t_2 = 0$$

$$t_3 = \sqrt{\frac{3}{5}} \approx 0{\cdot}774597$$

Thus, $$A_1+A_2+A_3 = 2$$

$$A_1t_1+A_2t_2+A_3t_3 = 0$$

$$A_1t_1^2+A_2t_2^2+A_3t_3^2 = \frac{2}{3}.$$

Substituting the values of t_1, t_2, t_3 and then solving, we obtain

$$A_1 = A_3 = \frac{5}{9}, A_2 = \frac{8}{9}$$

$$I = \int_{-1}^1 \frac{1}{1+x^2}dx$$

$$= \frac{5}{9}\cdot\frac{1}{1+(-0{\cdot}774597)^2} + \frac{8}{9}\cdot\frac{1}{1+0} + \frac{5}{9}\cdot\frac{1}{1+(0{\cdot}774597)^2}$$

$$= 0{\cdot}3472222 + 0{\cdot}8888888 + 0{\cdot}3472222$$

$$= 1{\cdot}583332.$$ **Ans.**

Q. 33. Evaluate the integral

$$I = \int_0^1 \frac{dx}{1+x}$$

using Gauss-Legendre three point formula. **(Indore 2022)**

Solution : Here $I = \int_0^1 \frac{dx}{1+x}$.

First we transform the interval [0, 1] to the interval [–1, 1]. Let

$$x = \frac{b-a}{2}t + \frac{b+a}{2},$$ **[where $a = 0$, $b = 1$]**

$$= \frac{1-0}{2}t + \frac{1+0}{2} = \frac{1}{2}t + \frac{1}{2}$$

so that $dx = \frac{1}{2}$.

The limits of integration becomes –1to 1.

$$\therefore \quad I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{dt}{t+3} = \sum_{i=1}^{3} w_i f(t_i)$$

$$= w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3).$$

Consider the Legendre polynomial of degree three

$$P_3(t) = \frac{1}{2}(5t^3 - 3t).$$

The zeros of $P_3(t)$ are given by $P_3(t) = 0$ *i.e.*,

$$\frac{1}{2}(5t^3 - 3t) = 0 \text{ } i.e., \text{ } t(5t^2 - 3) = 0.$$

$\therefore$ The roots of $P_3(t) = 0$ are $t_1 = -\sqrt{\frac{3}{5}}, t_2 = 0, t_3 = \sqrt{\frac{3}{5}}$ and

$$w_1 + w_2 + w_3 = 2$$

$$w_1 t_1 + w_2 t_2 + w_3 t_3 = 0$$

$$w_1 t_1^2 + w_2 t_2^2 + w_3 t_3^2 = \frac{2}{3}.$$

Substituting $t_1 = -\sqrt{\frac{3}{5}}, t_2 = 0$ and $t_3 = \sqrt{\frac{3}{5}}$ and solving, we obtain

$$w_1 = w_3 = \frac{5}{9}, w_2 = \frac{8}{9}.$$

Hence $$I = \left[\frac{5}{9}\left(\frac{1}{3-\sqrt{\frac{3}{5}}}\right) + \frac{8}{9}\left(\frac{1}{3+0}\right) + \frac{5}{9}\left(\frac{1}{3+\sqrt{\frac{3}{5}}}\right)\right]$$

$$= \frac{1}{9}\left[\frac{5}{2{\cdot}225403} + \frac{8}{3} + \frac{5}{3{\cdot}774597}\right]$$

$$= 0{\cdot}693122.$$

The exact solution is $I = 0{\cdot}693147$. **Ans.**

Q. 34. If $f(x)$ is a polynomial of degree 2 prove that :

$$\int_0^1 f(x)\,dx = \frac{1}{12}[5f(0) + 8f(1) - f(2)].$$ **(Indore 2013; Chhatarpur 2022)**

Solution : We have $\int_0^1 f(x)dx = \sum_{i=0}^{2} A_i f(x_i)$ ...(1)

It is clear that $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

Putting $f(x) = x^k$, $(k = 0, 1, 2)$, we get

$$I_0 = \int_0^1 x^0 dx = [x]_0^1 = 1$$

$$I_1 = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$I_2 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

Now $I_0 = 1 = \sum_{i=0}^{2} A_i = A_0 + A_1 + A_2$

$$I_1 = \frac{1}{2} = \sum_{i=0}^{2} A_i x_i = A_0 x_0 + A_1 x_1 + A_2 x_2 = A_0(0) + A_1(1) + A_2(2)$$

$$I_2 = \frac{1}{3} = \sum_{i=0}^{2} A_i x_i^2 = A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2 = A_0(0)^2 + A_1(1)^1 + A_2(2)^2$$

Solving the above set of three equations namely

$$A_0 + A_1 + A_2 = 1$$

$$A_1 + 2A_2 = \frac{1}{2}$$

$$A_1 + 4A_2 = \frac{1}{3}$$

We obtain $A_0 = \frac{5}{12}$, $A_1 = \frac{8}{12}$ and $A_2 = -\frac{1}{12}$

Substituting in eqn. (1), we get

$$\int_0^1 f(x)dx = \frac{5}{12}f(0) + \frac{8}{12}f(x) - \frac{1}{12}f(x)$$

$$= \frac{1}{12}[5f(0) + 8f(1) - f(2)].$$ **Proved.**

Objective Type Questions

1. Relation between E and Δ is : (Rewa 2022)

(a) $E = 1+\Delta$ (b) $E = 1-\Delta$

(c) $E = 1+\nabla$ (d) None of these.

2. The value of $\Delta \log x$ is : (Indore 2012, 13, 14; Bhopal 13, 16)

(a) $\log\left(1-\frac{h}{x}\right)$ (b) $\log\left(1+\frac{h}{x}\right)$ (c) $\log\frac{h}{2}$ (d) None of these.

3. The value of $\Delta\frac{f(x)}{g(x)}$ is : (Jabalpur2012)

(a) $\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x-h)g(x)}$ (b) $\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{\{g(x)\}^2}$

(c) $\frac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x+h)g(h)}$ (d) None of these.

4. The value of Δe^x : (Jabalpur 2012)

(a) $e^x(e^h+1)$ (b) $e^x(e^h-1)$

(c) $e^x(e^{x+h}+1)$ (d) $e^x(e^{x-h}-1)$.

5. The value of δ is : (Ujjain 2013)

(a) $\Delta E^{-1/2}$ (b) $\Delta E^{1/2}$ (c) $\nabla E^{-1/2}$ (d) None of these.

6. The value of central difference operator is : (Bhopal 2015)

(a) $E^{1/2}+E^{-1/2}$ (b) $\frac{1}{2}(E^{1/2}-E^{-1/2})$

(c) $E^{1/2}-E^{-1/2}$ (d) None of these.

7. The value of mean operator μ is :

(a) $\frac{1}{2}(E^{1/2}-E^{-1/2})$ (b) $\frac{1}{2}(E^{1/2}+E^{-1/2})$

(c) $E^{1/2}+E^{-1/2}$ (d) None of these.

8. Newton's forward interpolation formula gives the better result when the value of x corresponding to $f(x)$ is :

(a) Near the beginning of tabulated values

(b) Near the ending of tabulated values

(c) Near the mid value of tabulated values

(d) None of these.

9. The required polynomial for $f(0)=1, f(1)=3, f(3)=55$ using Newton's divided difference interpolation formula is :

(a) $8x^2-6x+1$ (b) x^2-6x+1

(c) $8x^2+6x+1$ (d) None of these.

10. If $\pi(x)=(x-x_0)(x-x_1).....(x-x_n)$ and $\pi'(x)=\frac{d}{dx}\pi(x)$, then Lagrange's interpolation formula is :

(a) $P_n(x)=\sum_{i=0}^{n}\frac{\pi(x)f(x_i)}{(x-x_i)\pi'(x_i)}$ (b) $P_n(x)=\sum_{i=0}^{n}\frac{\pi'(x_i)f(x_i)}{(x-x_i)\pi(x)}$

(c) $P_n(x)=\sum_{i=0}^{n}\frac{\pi(x)}{(x-x_i)\pi'(x_i)}$ (d) None of these.

11. The value of $\Delta \log f(x)$ is : (Indore 2013)

(a) $\log\left[1-\frac{\Delta f(x)}{f(x)}\right]$ (b) $\log\left[1+\frac{f(x)}{\Delta f(x)}\right]$

(c) $\log\left[1+\frac{\Delta f(x)}{f(x)}\right]$ (d) None of these.

12. The sum of lagrangean coefficient is : (Indore 2011, 15, 17; Jabalpur 14, 15, 16, 17; Rewa 16; Bhopal 17)

(a) $n!$ (b) 1 (c) -1 (d) None of these.

13. In $\Delta f(x)=f(x+h)-f(x)$, Δ is : (Ujjain 2011)

(a) Forward difference operator (b) Backward difference operator

(c) First difference operator (d) Modulus operator.

14. Lagrange's formula can be used for interopolation, when interval of values of x_i is : (Rewa 2011)

(a) Equal (b) Equal or unequal (c) unequal (d) None of these.

15. The n^{th} divided difference of a polynomial of n^{th} degree is always a function of variable. Its degree is : (Rewa 2011)

(a) 1 (b) 0 (c) -1 (d) More than one.

16. The value of $\nabla f(x_k)$ in backward difference is : (Bhopal 2011)

(a) $f(x_{k-h})-f(x_k)$ (b) $f(x_{0+h})-f(x_0)$

(c) $f(x_k)-f(x_{k-h})$ (d) $f(x_k)-f(x_{k+h})$.

17. The formula to find the differentiation of any function at any point in middle of the table :

(a) Any central formula

(b) Newton's forward difference formula

(c) Newton backward difference formula

(d) None of these.

18. If arguments are not equidistance. Then the formula to find the differentiation of a function of any point :

(a) Newton forward difference formula

(b) Newton backward difference formula

(c) Newton divided difference formula

(d) None of these.

19. The value of any divided difference is independent on :
(a) Argument (b) Matrix (c) Both (d) None of these.

20. ∇^2 is equal to : **(Bhopal 2014)**
(a) $(1-E^{-1})^2$ (b) $(1+E^{-1})^2$ (c) $(E-1)^2$ (d) None of these.

21. If differencing interval is h then the value of $\Delta\frac{1}{f(x)}$ is :
(a) $\frac{\Delta f(x)}{f(x)f(x+h)}$ (b) $-\frac{\Delta f(x)}{f(x)f(x+h)}$
(c) $\frac{\Delta^2 f(x)}{f(x)f(x+h)}$ (d) None of these.

22. n^{th} divided difference can be represented as the division of two determinates of order : **(Rewa 2016)**
(a) n (b) $(n-1)$ (c) $(n+1)$ (d) n !.

23. The value of ∇^3 is : **(Jabalpur 2016)**
(a) $(1-E^{-1})^3$ (b) $(1+E)^3$ (c) $(E-1)^2$ (d) $(E+1)$.

24. In forward difference, value of $\Delta f(x_k)$ is : **(Indore 2017)**
(a) $f(x_k-h)-f(x_k)$ (b) $f(x_k+h)-f(x_k)$
(c) $f(x_k)-f(x_k-h)$ (d) $f(x_k)-f(x_k+h)$.

25. Value of $f'(a)$ by using Newton Backward difference interpolation is : **(Bhopal 2017)**
(a) $\frac{1}{h}\left[\nabla f(a)+\frac{1}{2}\nabla^2 f(a)+\frac{1}{3}\nabla^3 f(a)+\frac{1}{4}\nabla^4 f(a)+.....\right]$
(b) $\frac{1}{h}\left[\nabla f(a)-\frac{1}{2}\nabla^2 f(a)+\frac{1}{3}\nabla^3 f(a)-\frac{1}{4}\nabla^4 f(a)+.....\right]s$
(c) $\frac{1}{h}\left[1+\nabla f(a)+\frac{1}{2}\nabla^2 f(a)+\frac{1}{3}\nabla^3 f(a)+\frac{1}{4}\nabla^4 f(a)+.....\right]$
(d) None of these.

26. The relation between Δ and ∇ is : **(Indore 2022)**
(a) $\nabla\Delta=\Delta-\nabla$ (b) $(1+\Delta)(1-\nabla)=1$
(c) (a) and (b) both (d) None of these.

27. The relation among E, Δ and ∇ : **(Chhindwara 2022)**
(a) $E\Delta=\Delta E=\nabla+E$ (b) $\Delta E=E\Delta=\nabla-E$
(c) $E\nabla=\nabla E=\nabla$ (d) $\Delta\nabla=E+1$.

Ans. 1. (a), 2. (b), 3. (b), 4. (c), 5. (a), 6. (c), 7. (a), 8. (a), 9. (a), 10. (a), 11. (c), 12. (b), 13. (a), 14. (c), 15. (b), 16. (c), 17. (a), 18. (c), 19. (c), 20. (a), 21. (b), 22. (c), 23. (a), 24. (b), 25. (a), 26. (c), 27. (c).

●●

UNIT 3

Short and Long Answer Type Questions

Q. 1. Solve the following system of equations by Gauss elimination method
$2x-y+3z=9,\ x+y+z=6,\ x-y+z=2.$
(Jabalpur 2011, 14, 17; Ujjain 14, 22; Indore 16, 22; Gwalior 22)

Solution : The given system of equations can be expressed in matrix form as :

$$AX=B$$

Where $A=\begin{bmatrix}2&-1&3\\1&1&1\\1&-1&1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}9\\6\\2\end{bmatrix}$

$\therefore$ Augmented matrix, $[A:B]=\begin{bmatrix}2&-1&3&:&9\\1&1&1&:&6\\1&-1&1&:&2\end{bmatrix}$

Now, taking $a_{11}=2$ as pivot, operating $R_{21}\left(-\frac{1}{2}\right)$ and $R_{31}\left(-\frac{1}{2}\right)$, we get

$$[A:B]\sim\begin{bmatrix}2&-1&3&:&9\\0&3/2&-1/2&:&3/2\\0&-1/2&-1/2&:&-5/2\end{bmatrix}$$

Taking $b_{22}=\frac{3}{2}$ as pivot, operating $R_{32}\left(\frac{1}{3}\right)$, we get

$$[A:B]\sim\begin{bmatrix}2&-1&3&:&9\\0&3/2&-1/2&:&3/2\\0&0&-2/3&:&-2\end{bmatrix}$$

$$\Rightarrow\quad\begin{bmatrix}2&-1&3\\0&3/2&-1/2\\0&0&-2/3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\3/2\\-2\end{bmatrix}$$

$$\Rightarrow\quad\left.\begin{aligned}2x-y+3z&=9\\(3/2)y-(1/2)z&=3/2\\-(2/3)z&=-2\end{aligned}\right\}$$

Using back substitution, we get $z=3,\ y=2, x=1$

Hence, the required solution is $x=1,\ y=2,\ z=3$. **Ans.**

Q. 2. Solve $10x-7y+3z+5u=6, -6x+8y-z-4u=5, 3x+y+4z+11u=2,$ $5x-9y-2z+4u=7$, by Gauss's elimination method. **(Indore 2014)**

Solution : Dividing first equation of the given system by the pivot element by 10, we have,

$$x - 0{\cdot}7y + 0{\cdot}3z + 0{\cdot}5u = 0{\cdot}6 \quad ...(1)$$

$$-6x + 8y - z - 4u = 5 \quad ...(2)$$

$$3x + y + 4z + 11u = 2 \quad ...(3)$$

$$5x - 9y - 2z + 4u = 7 \quad ...(4)$$

Step I. To eliminate x, apply eqn. (2) + 6 eqn. (1), eqn. (3) – 3 eqn. (1) and eqn. (4) – 5 eqn. (1), we get

$$3{\cdot}8y + 0{\cdot}8z - u = 8{\cdot}6 \quad ...(5)$$

$$3{\cdot}1y + 3{\cdot}1z + 9{\cdot}5u = 0{\cdot}2 \quad ...(6)$$

$$-5{\cdot}5y - 3{\cdot}5z + 1{\cdot}5u = 4 \quad ...(7)$$

Step II. To eliminate y, apply 3·8 eqn. (6) – 3·1 eqn. (5) and 3·8 eqn. (7) + 5·5 eqn. (5), we get

$$9{\cdot}3z + 39{\cdot}2u = -25{\cdot}9 \quad ...(8)$$

$$-8{\cdot}9z + 0{\cdot}2u = 62{\cdot}5 \quad ...(9)$$

Step III. To eliminate z, apply 9·3 eqn. (9) + 8·9 eqn. (8), we get

$$350{\cdot}74u = 350{\cdot}74 \Rightarrow u = 1$$

Step IV. By back substitution

From eqn. (8), we get $z = -7$

From eqn. (5), we get $y = 4$

Lastly from eqn. (1), $x = 0{\cdot}6 + 0{\cdot}7 \times 4 + 0{\cdot}3 \times 7 - 0{\cdot}5 \times 1 = 5$

Hence, the solution of the given system is $x = 5, y = 4, z = -7, u = 1$. **Ans.**

Q. 3. Write short note on Gauss elimination method and solve following system of equations by this method :

$6x_1 + 3x_2 + 2x_3 = 6$, $6x_1 + 4x_2 + 3x_3 = 0$ and $20x_1 + 15x_2 + 12x_3 = 0$.

(Ujjain 2011; Jabalpur 12, 15; Indore 13; Rewa 14, 22; Gwalior 16; Bhopal 17)

Solution : In this method, system of equations are reduced to equivalent upper triangular system whose solution is obtained by back substitution.

Given system of equations :

	Check sum	
$6x_1 + 3x_2 + 2x_3 = 6$	17	...(1)
$6x_1 + 4x_2 + 3x_3 = 0$	13	...(2)
$20x_1 + 15x_2 + 12x_3 = 0$	47	...(3)

Step 1 : To eliminate x_1, apply eqn. (2) – eqn. (1) and eqn. (3) – $\frac{20}{6}$ × eqn. (1), we get

	Check sum	
$6x_1 + 3x_2 + 2x_3 = 6$	17	
$x_2 + x_3 = -6$	-4	...(4)
$5x_2 + \frac{16}{3}x_3 = -20$	$-\frac{29}{3}$	...(5)

Step 2 : To eliminate x_2, apply eqn. (5) – 5 × eqn. (4), we get

	Check sum	
$6x_1 + 3x_2 + 2x_3 = 6$	17	
$x_2 + x_3 = -6$	-4	
$\frac{1}{3}x_3 = 10$	$\frac{31}{3}$	...(6)

Step 3 : By back substitution, from eqn. (6) $x_3 = 30$

On putting value of x_3 in eqn. (4), we get $x_2 = -36$

On putting value of x_2 and x_3 in eqn. (1), we get

$$x_1 = \frac{1}{6}[6 + 108 - 60] = 9$$

Hence, the solution of the given system is

$$x_1 = 9, x_2 = -36, x_3 = 30.$$ **Ans.**

Q. 4. Solve the equation $x + 2y + z = 3$; $2x + 3y + 3z = 10$; $3x - y + 2z = 13$ by Gauss-Jordan method. **(Ujjain 2016; Bhopal 22)**

Solution : The given equation is equal to :

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right], \quad [\text{Operation } R_2 + (-2)R_1 \Rightarrow R_{21}(-2)$$

$$R_3 + (-3)R_1 \Rightarrow R_{31}(-3)]$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right], \quad [\text{Operation } R_{32}(-7)]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right], \quad [\text{Operation } R_{12}(2)]$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right], \quad \left[\text{Operation } R_3\left(\frac{1}{8}\right)\right]$$

$$[A:B] \sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right], \quad [\text{Operation } R_{13}(3), R_{23}(1)]$$

$\Rightarrow$ $x = 2, y = -1, z = 3.$ **Ans.**

Q. 5. Solve the following system of equation by using :

$10x + y + 2z = 13, 3x + 10y + z = 14, 2x + 3y + 10z = 15.$

(Jabalpur 2011, 16; Bhopal 16; Rewa 22)

Solution : The given system of equations in matrix form :

$$AX = B$$

Where $$A = \begin{bmatrix} 10 & 1 & 2 \\ 3 & 10 & 1 \\ 2 & 3 & 10 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 13 \\ 14 \\ 15 \end{bmatrix}$$

$\therefore$ Augmented matrix, $[A:B] = \begin{bmatrix} 10 & 1 & 2 & : & 13 \\ 3 & 10 & 1 & : & 14 \\ 2 & 3 & 10 & : & 15 \end{bmatrix}$

Taking $a_{11} = 10$ as pivot, operating $R_{21}\left(-\frac{3}{10}\right)$ and $R_{31}\left(-\frac{1}{5}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 10 & 1 & 2 & : & 13 \\ 0 & 97/10 & 2/5 & : & 101/10 \\ 0 & 14/5 & 48/5 & : & 62/5 \end{bmatrix}$$

Taking $b_{22} = \frac{97}{10}$ as pivot, operating $R_{32}\left(-\frac{28}{97}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 10 & 1 & 2 & : & 13 \\ 0 & 97/10 & 2/5 & : & 101/10 \\ 0 & 0 & 920/97 & : & 920/97 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 10 & 1 & 2 \\ 0 & 97/10 & 2/5 \\ 0 & 0 & 920/97 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 101/10 \\ 920/97 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 10x + y + 2z &= 13 \\ (97/10)y + (2/5)z &= 101/10 \\ (920/97)z &= 920/97 \end{aligned}\right\}.$$

By back substitution, we get

$$z = 1, y = 1, x = 1$$

Hence the required solution is $x = 1, y = 1, z = 1$. **Ans.**

Q. 6. Solve the equations by using Gauss-Jordan method :

$x + 4y - z = -5, x + y - 6z = -12, 3x - y - z = 4.$ **(Ujjain 2017; Rewa 22)**

Solution : The given system of equations in matrix form can be written as :

$$AX = B$$

where $$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$$

The augmented matrix as follows

$$[A:B] = \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 1 & 1 & -6 & : & -12 \\ 3 & -1 & -1 & : & 4 \end{bmatrix}$$

Now taking $a_{11} = 1$ as pivot, operating $R_{21}(-1)$ and $R_{31}(-3)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & -3 & -5 & : & -7 \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$

Applying $R_2\left(-\frac{1}{3}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 4 & -1 & : & -5 \\ 0 & 1 & \frac{5}{3} & : & \frac{7}{3} \\ 0 & -13 & 2 & : & 19 \end{bmatrix}$$

Taking $b_{22} = 1$ as pivot, operating R_{21} (–4) and R_{32} (13), we get

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -\frac{23}{3} & : & -\frac{43}{3} \\ 0 & 1 & \frac{5}{3} & : & \frac{7}{3} \\ 0 & 0 & \frac{71}{3} & : & \frac{148}{3} \end{bmatrix}$$

Applying $R_3\left(\frac{3}{71}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -\frac{23}{3} & : & -\frac{43}{3} \\ 0 & 1 & \frac{5}{3} & : & \frac{7}{3} \\ 0 & 0 & 1 & : & \frac{148}{71} \end{bmatrix}$$

Taking $C_{33} = 1$ as pivot, operating $R_{13}\left(\frac{23}{3}\right)$ and $R_{23}\left(-\frac{5}{3}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{117}{71} \\ 0 & 1 & 0 & : & -\frac{81}{71} \\ 0 & 0 & 1 & : & \frac{148}{71} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{117}{71}\\-\frac{81}{71}\\\frac{148}{71}\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}\frac{117}{71}\\-\frac{81}{71}\\\frac{148}{71}\end{bmatrix}. \qquad \text{Ans.}$$

Q. 7. (a) Solve the following system of 3 × 3 by Gauss Elimination method :

$$3x+6y+z=16, 2x+4y+3z=13, x+3y+2z=9.$$

(Rewa 2011, 15; Ujjain 16; Chhatarpur 22)

Solution : Here $A=\begin{bmatrix}3&6&1\\2&4&3\\1&3&2\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}16\\13\\9\end{bmatrix}$

Given system of equation in matrix form

$$AX=B$$

Augmented matrix, $[A:B]=\left[\begin{array}{ccc:c}3&6&1&16\\2&4&3&13\\1&3&2&9\end{array}\right]$

By $R_2 \Leftrightarrow R_3$

$$[A:B]\sim\left[\begin{array}{ccc:c}3&6&1&16\\1&3&2&9\\2&4&3&13\end{array}\right]$$

Now $R_2 \to R_2-\frac{1}{3}R_1$ and $R_3 \to R_3-\frac{2}{3}R_1$

$$[A:B]=\left[\begin{array}{ccc:c}3&6&1&16\\0&1&\frac{5}{3}&\frac{11}{3}\\0&0&\frac{7}{3}&\frac{7}{3}\end{array}\right]$$

$$\Rightarrow \quad 3x+6y+z=16$$

$$y+\frac{5}{3}z=\frac{11}{3}$$

$$\frac{7}{3}z=\frac{7}{3}\Rightarrow z=1$$

$$\Rightarrow \quad y+\frac{5}{3}=\frac{11}{3}\Rightarrow y=\frac{11}{3}-\frac{5}{3}=2$$

and $3x+6.2+1=16$

$$\Rightarrow \quad 3x=3\Rightarrow x=1.$$

Ans.

Q. 7. (b) Solve by Gauss elimination method :

$$2x+y+z=10, 3x+2y+3z=18, x+4y+9z=16.$$

(Rewa 2016)

Solution : Solve as Q. No. 7 (a).

Q. 8. Solve by Gauss-elimination method :

$$2x+y+4z=12, 8x-3y+2z=23, 4x+11y-z=33.$$

(Rewa 2017; Bhopal 17)

Solution : Given system of equations in matrix form :

$$AX=B$$

Where

$$\therefore \text{ Augmented matrix, } [A:B]=\left[\begin{array}{ccc:c}2&1&4&12\\8&-3&2&23\\4&11&-1&33\end{array}\right]$$

Taking $a_{11}=2$ as pivot, operating $R_{21}(-4)$ and $R_{31}(-2)$, we get

$$[A:B]\sim\left[\begin{array}{ccc:c}2&1&4&12\\0&-7&-14&-25\\0&9&-9&9\end{array}\right]$$

Taking $b_{22}=-7$ as pivot, operating $R_{32}\left(\frac{9}{7}\right)$, we get

$$[A:B]\sim\left[\begin{array}{ccc:c}2&1&4&12\\0&-7&-14&-25\\0&0&-27&-\frac{162}{7}\end{array}\right]$$

$$\Rightarrow \quad \left.\begin{aligned}2x+y+4z&=12\\-7y-14z&=-25\\-27z&=-\frac{162}{7}\end{aligned}\right\}.$$

Using back substitution, we get

$$z = \frac{6}{7} = 0{\cdot}8571, y = 1{\cdot}8571, x = 3{\cdot}3573$$

Hence, the required solution is

$$x = 3{\cdot}3573, y = 1{\cdot}8571, z = 0{\cdot}8571.$$ Ans.

Q. 9. Solve by Gauss-elimination method : (Indore 2011)

$$x_1 + x_2 = 2;\ 2x_1 + 3x_2 = 5.$$

Solution : The given system of the given equations :

$$AX = B$$

Where $$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

Augmented metrix

$$[A : B] = \begin{bmatrix} 1 & 1 & : & 2 \\ 2 & 3 & : & 5 \end{bmatrix}$$

Operation by, $R_2 \to R_2 - 2R_1$

$$[A : B] = \begin{bmatrix} 1 & 1 & : & 2 \\ 0 & 1 & : & 1 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = 2$$
$$x_2 = 1$$

Using back substitution, we get

$$x_2 = 1, x_1 = 1$$

Hence the required solution is :

$$x_1 = 1, x_2 = 1.$$ Ans.

Q. 10. Solve the following system of equations by *LU* factorization method :

$$x_1 + 2x_2 + 3x_3 = 14$$
$$2x_1 + 5x_2 + 2x_3 = 18$$
$$3x_1 + x_2 + 5x_3 = 20.$$

(Gwalior 2011; Indore 12; Ujjain 13, 15; Bhopal 11, 13; Chhatarpur 22)

Solution : $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$A = LU \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Step 1 : On comparing the elements of first row,

$$u_{11} = 1,\ u_{12} = 2,\ u_{13} = 3$$

and on comparing remaining elements of first column,

$$l_{21}u_{11} = 2,\ l_{31}u_{11} = 3$$

$$\Rightarrow \quad l_{21} = 2,\ l_{31} = 3 \qquad [\because u_{11} = 1]$$

Step 2 : On comparing remaining elements of second row,

$$l_{21}u_{12} + u_{22} = 5,\ l_{21}u_{13} + u_{23} = 2$$

$$\Rightarrow \quad u_{22} = 5 - 2\times 2 = 1$$

and $$u_{23} = 2 - 2\times 3 = -4$$

and on comparing remaining elements of second column,

$$l_{31}u_{12} + l_{32}u_{22} = 1$$

$$\Rightarrow \quad l_{32} = \frac{1 - 3\times 2}{u_{22}} = \frac{1-6}{1} = -5$$

Step 3 : On comparing remaining elements of third row,

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 5$$

$$\Rightarrow \quad u_{33} = 5 - 3\times 3 - (-5)(-4) = -24.$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix},\ U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix}$$

Step 4 : Now matrix equation $AX = B$ ...(1)

$$\Rightarrow \ LUX = B \ \Rightarrow \ LC = B \quad \text{...(2)}$$

Where $C = UX$ ...(3)

From eqn. (2), $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

From forward substitution $c_1 = 14$,

$$2c_1 + c_2 = 18$$

$$\Rightarrow \quad c_2 = 18 - 2\times 14 = -10$$

$$3c_1 - 5c_2 + c_3 = 20$$

$$\Rightarrow \quad c_3 = 20 - 3\times 14 + 5(-10) = -72$$

Now, from eqn. (3), $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

From backward substitution $x_3 = \frac{-72}{-24} = 3$

$$x_2 - 4x_3 = -10$$

$$\Rightarrow \quad x_2 = -10 + 4\times 3 = 2$$

$$x_1 + 2x_2 + 3x_3 = 14$$

$$\Rightarrow \quad x_1 = 14 - 2\times 2 - 3\times 3 = 1$$

Hence, solution set is (1, 2, 3). Ans.

Q. 11. Solve the following system of equations by LU decomposition method

$$x+5y+z=14;\ 2x+y+3z=13;\ 3x+y+4z=17.$$ **(Indore 2016)**

Solution : The given system of equation is

$$AX=B$$

Where $A=\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, B=\begin{bmatrix}14\\13\\17\end{bmatrix}$

Let $A=LU=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$

i.e., $\begin{bmatrix}1&5&1\\2&1&3\\3&1&4\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$

Then, we have

(i) $u_{11}=1, u_{12}=5, u_{13}=1$

(ii) $l_{21}u_{11}=2, \quad \therefore \ l_{21}=2,$

$l_{31}u_{11}=3, \quad \therefore \ l_{31}=3,$

(iii) $l_{21}u_{12}+u_{22}=1, \quad \therefore \ u_{22}=-9,$

$l_{21}u_{13}+u_{23}=3, \quad \therefore \ u_{23}=1,$

(iv) $l_{31}u_{12}+l_{32}u_{22}=1 \quad \therefore \ l_{32}=\frac{14}{9},$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=4 \quad \therefore \ u_{33}=-\frac{5}{9},$

Thus, $A=\begin{bmatrix}1&0&0\\2&1&0\\3&\frac{14}{9}&1\end{bmatrix}\begin{bmatrix}1&5&1\\0&-9&1\\0&0&-\frac{5}{9}\end{bmatrix}$

Writing $UX=V$ where $V=\begin{bmatrix}u\\v\\w\end{bmatrix}$, then the given system becomes

$$A=\begin{bmatrix}1&0&0\\2&1&0\\3&\frac{14}{9}&1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}14\\13\\17\end{bmatrix}$$

$$\Rightarrow \quad u=14, 2u+v=13, 3u+\frac{14}{9}v+w=17$$

Solving these equations, we get

$$u=14, v=-15, w=-\frac{5}{3}$$

Hence, the original system becomes

$$\begin{bmatrix}1&5&1\\0&-9&1\\0&0&-\frac{5}{9}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}14\\-15\\-\frac{5}{3}\end{bmatrix}$$

$$x+5y+z=14, -9y+z=-15, -\frac{5}{9}z=-\frac{5}{3}$$

By back substitution, we have

$$z=3, y=2, x=1.$$

$\therefore$ The required solution is $x=1, y=2, z=3.$ **Ans.**

Q. 12. Apply LU decomposition method to solve the equations :

$3x+2y+7z=4; 2x+3y+z=5, 3x+4y+z=7.$

(Jabalpur 2011, 14, 17; Indore 13, 22; Ujjain 14, 17, 22; Bhopal 16; Gwalior 17; Chhindwara 22)

Solution : Here $A=\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}$

Let $A=LU$ *i.e.*, $\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$

Comparing corresponding elements,

$$A=\begin{bmatrix}3&2&7\\2&3&1\\3&4&1\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Then, we have

(i) $u_{11}=3, u_{12}=2, u_{13}=7$

(ii) $l_{21}u_{11}=2 \Rightarrow l_{21}=\frac{2}{3}$

$l_{31}u_{11}=3 \Rightarrow l_{31}=1$

(iii) $l_{21}u_{12}+u_{22}=3 \quad \Rightarrow u_{22}=\frac{5}{3}$

$l_{21}u_{13}+u_{23}=1 \quad \Rightarrow u_{23}=-\frac{11}{3}$

(iv) $l_{31}u_{12}+l_{32}u_{22}=4 \quad \Rightarrow l_{32}=\frac{6}{5}$

(v) $l_{31}u_{13}+l_{32}u_{23}+u_{33}=1 \Rightarrow u_{33}=-\frac{8}{5}$

Thus $$A=\begin{bmatrix}1&0&0\\2/3&1&0\\1&6/5&1\end{bmatrix}\begin{bmatrix}3&2&7\\0&5/3&-11/3\\0&0&-8/5\end{bmatrix}$$

Writing $UX=V$, where $V=\begin{bmatrix}u\\v\\w\end{bmatrix}$ then the given system becomes

$$=\begin{bmatrix}1&0&0\\2/3&1&0\\1&6/5&1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}4\\5\\7\end{bmatrix}$$

$$\Rightarrow \quad u=4, \frac{2}{3}u+v=5, u+\frac{6}{5}v+w=7$$

Solving these equations, we get $u=4, v=\frac{7}{3}, w=\frac{1}{5}$

Hence the original system becomes

$$\begin{bmatrix}3&2&7\\0&5/3&-11/3\\0&0&-8/5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\7/3\\1/5\end{bmatrix}$$

$$\Rightarrow \quad 3x+2y+7z=4, \frac{5}{3}y-\frac{11}{3}z=\frac{7}{3}, -\frac{8}{5}z=\frac{1}{5}$$

By back substitution, we have $z=-\frac{1}{8}, y=\frac{9}{8}$ and $x=\frac{7}{8}$.

$\therefore$ The required solution is $x=\frac{7}{8}, y=\frac{9}{8}, z=-\frac{1}{8}$. **Ans.**

Q. 13. Solve the system of equations

$3x_1+2x_2+x_3=10$; $2x_1+3x_2+2x_3=14$; $x_1+2x_2+3x_3=14$.
by using *LU* decomposition method.

(Ujjain 2011; Jabalpur 12, 15; Indore 14, 17; Gwalior 16)

Solution : The given system of equations is

$$AX=B, \text{ where } A=\begin{bmatrix}3&2&1\\2&3&2\\1&2&3\end{bmatrix}$$

$$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \text{ and } B=\begin{bmatrix}10\\14\\14\end{bmatrix}$$

Let $A=LU=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}\begin{bmatrix}u_{11}&u_{12}&u_{13}\\0&u_{22}&u_{23}\\0&0&u_{33}\end{bmatrix}$

$$i.e., \begin{bmatrix}3&2&1\\2&3&2\\1&2&3\end{bmatrix}=\begin{bmatrix}u_{11}&u_{12}&u_{13}\\l_{21}u_{11}&l_{21}u_{12}+u_{22}&l_{21}u_{13}+u_{23}\\l_{31}u_{11}&l_{31}u_{12}+l_{32}u_{22}&l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

Then we have

(i) $u_{11}=3 \qquad u_{12}=2 \qquad u_{13}=1$

(ii) $l_{21}u_{11}=2 \Rightarrow l_{21}=\frac{2}{3}$

$l_{31}u_{11}=1 \Rightarrow l_{31}=\frac{1}{3}$

(iii) $l_{21}u_{12}+u_{22}=3 \Rightarrow u_{22}=3-\left(\frac{2}{3}\right)2=\frac{5}{3}$

$l_{21}u_{13}+u_{23}=2 \Rightarrow u_{23}=2-\left(\frac{2}{3}\right)1=\frac{4}{3}$

(iv) $l_{31}u_{12}+l_{32}u_{22}=2 \Rightarrow l_{32}=\left[2-\left(\frac{1}{3}\right)2\right]\Big/\left(\frac{5}{3}\right)=\left(\frac{4}{5}\right)$

$l_{31}u_{13}+l_{32}u_{23}+u_{33}=3 \Rightarrow u_{33}=3-\left(\frac{1}{3}\right)1-\left(\frac{4}{5}\right)\left(\frac{4}{3}\right)=\frac{24}{15}$

Thus we have

$$L=\begin{bmatrix}1&0&0\\2/3&1&0\\1/3&4/5&1\end{bmatrix} \text{ and } U=\begin{bmatrix}3&2&1\\0&5/3&4/3\\0&0&24/15\end{bmatrix}$$

Writing $UX=V=\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$ then the given system becomes

$LV=B$

$$\Rightarrow \begin{bmatrix}1&0&0\\2/3&1&0\\1/3&4/5&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}10\\14\\14\end{bmatrix} \Rightarrow v_1=10, \frac{2}{3}v_1+v_2=14,$$

$$\frac{1}{3}v_1+\frac{4}{5}v_2+v_3=14$$

Solving these equations by forward substitution,

$$v_1=10, v_2=\frac{22}{3}, v_3=\frac{72}{15}$$

Hence, the original system becomes

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 72/15 \end{bmatrix}$$

$$i.e. \quad 3x_1 + 2x_2 + x_3 = 10, \frac{5}{3}x_2 + \frac{4}{3}x_3 = \frac{22}{3}, \frac{24}{15}x_3 = \frac{72}{15}$$

Solving these equations by back substitution, we get

$$x_3 = 3, x_2 = 2, x_1 = 1$$

$\therefore$ The required solution is $x_1 = 1, x_2 = 2$ and $x_3 = 3$. Ans.

Q. 14. Solve the following equation by *LU* decomposition method. (Rewa 2016)

$$2x + y + z = 2;\ x + 3y + 2z = 2;\ 3x + y + 2z = 2.$$

Solution :

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix}$$

Let $A = LU$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

$$\Rightarrow \quad u_{11} = 2, u_{12} = 1, u_{13} = 1$$

$$l_{21}u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$$

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow \frac{1}{2} \times 1 + u_{22} = 3 \Rightarrow u_{22} = 3 - \frac{1}{2} = \frac{5}{2}$$

$$l_{21}u_{13} + u_{23} = 2 \Rightarrow \frac{1}{2} \times 1 + u_{23} = 2 \Rightarrow u_{23} = \frac{3}{2}$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31}.2 = 3 \Rightarrow l_{31} = \frac{3}{2}$$

$$l_{31}.u_{12} + l_{32}u_{22} = 1 \Rightarrow \frac{3}{2}.1 + l_{32}.\frac{5}{2} = 1 \Rightarrow l_{32} = -\frac{1}{5}$$

$$l_{31}.u_{13} + l_{32}.u_{23} + u_{33} = 2 \Rightarrow \frac{3}{2} \times 1 + \left(-\frac{1}{5}\right) \times \frac{3}{2} + u_{33} = 2 \Rightarrow u_{33} = \frac{4}{5}$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -\frac{1}{5} & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{4}{5} \end{bmatrix}$$

$$LV = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -\frac{1}{5} & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad u = 2, \frac{1}{2}u + v = 2, \frac{3}{2}u - \frac{1}{5}v + w = 2$$

$$\Rightarrow \quad u = 2, v = 2, w = -\frac{3}{5}$$

$$V = \begin{bmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{bmatrix}$$

Now,

$$UX = V$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & \frac{4}{5} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -\frac{3}{5} \end{bmatrix}$$

$$\Rightarrow \quad 2x + y + z = 2$$

$$0 + \frac{5}{2}y + \frac{3}{2}z = 2$$

$$\frac{4}{5}z = -\frac{3}{5} \Rightarrow z = -\frac{3}{4}$$

$$y = \frac{5}{4}, x = \frac{3}{4}.$$ Ans.

Q. 15. Explain *LU* decomposition method. (Rewa 2017)

Solution : Direct Method : Triangularization method (*LU* decomposition method)

A system of linear equations can be solved by writing the coefficient matrix A as a product of a lower triangular matrix L and a upper triangular matrix U. Consider the system of equations :

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \ldots(1)$$

This system is equivalent to $AX = B$ ...(2)

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

and an upper triangular matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \text{ so that } A = LU$$

Also $LUX = B$ ...(3)

Let $UX = Y$ ...(4)

and hence $LY = B$ ...(5)

That is, $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ ...(6)

$\therefore \quad y_1 = b_1, l_{21}y_1 + y_2 = b_2, l_{31}y_1 + l_{32}y_2 + y_3 = b_3.$

By forward substitution, y_1, y_2, y_3 can be found our if L is known. From eqn. (4)

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$
$$u_{22}x_2 + u_{23}x_3 = y_2$$
$$u_{33}x_3 = y_3$$

From these x_1, x_2 and x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.

Now, L and U can be found from $LU = A$

i.e.,
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

i.e.,
$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknown these 9 equations, we can solve for 3l's and 6 u's.

That is, L and U are known. Hence X is found out. Going into details, we get a $u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$. That is the elements in the first row of U are same as the elements first of A.

Also,

$$l_{21}u_{11} = a_{21}, l_{21}u_{12} + u_{22} = a_{22}, l_{21}u_{13} + u_{23} = a_{23}$$

$$\therefore \quad l_{21} = \frac{a_{21}}{a_{11}}, u_{22} = a_{22} - \frac{a_{21}}{a_{11}}.a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}}.a_{13}$$

Again, $l_{31}u_{11} = a_{31}, l_{31}u_{12} + l_{32}u_{22} = a_{32}$ and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{23}$

Solving,

$$l_{31} = \frac{a_{31}}{a_{11}}, l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}}$$

$$u_{33} = a_{33} - \frac{a_{31}}{a_{11}}.a_{13} - \left(\frac{a_{32} - \frac{a_{31}}{a_{11}}.a_{12}}{a_{22} - \frac{a_{21}}{a_{11}}.a_{12}} \right) \left(a_{23} - \frac{a_{21}}{a_{11}}.a_{13} \right)$$

Therefore L and U are known. **Ans.**

Q. 16. Solve the following equations by using LU decomposition method

$$5x - 2y + z = 4; 7x + y - 5z = 8; 3x + 7y + 4z = 10.$$ **(Rewa 2017, 22)**

Solution :

$$\begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 10 \end{bmatrix}$$ ...(1)

Here, $A.X = B$

Let $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 5 & -2 & 1 \\ 7 & 1 & -5 \\ 3 & 7 & 4 \end{bmatrix}$$

$$\Rightarrow \quad u_{11} = 5, u_{12} = -2, u_{13} = 1$$

$$l_{21}u_{11} = 7, l_{21}u_{12} + u_{22} = 1, l_{21}u_{13} + u_{23} = -5$$

$$\Rightarrow \quad l_{21}=\frac{7}{5}, u_{22}=\frac{19}{5}, u_{23}=\frac{-32}{5}$$

and $\quad l_{31}u_{11}=3, l_{31}u_{12}+l_{32}u_{22}=7, l_{31}u_{13}+l_{32}u_{23}+u_{33}=4$

$$\Rightarrow \quad l_{31}=\frac{3}{5}, l_{32}=\frac{41}{19}, u_{31}=\frac{327}{19}$$

$\because \quad LUX=B$

Hence, $LY=B$, where $UX=Y$

From $LY=B$,

$$\begin{bmatrix}1 & 0 & 0\\ 7/5 & 1 & 0\\ 3/5 & 41/19 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}4\\ 8\\ 10\end{bmatrix}$$

$$\Rightarrow \quad y_1=4, \frac{7}{5}y_1+y_2=8, \frac{3}{5}y_1+\frac{41}{19}y_2+y_3=10$$

$$\Rightarrow \quad y_2=\frac{12}{5}, y_3=\frac{46}{19}$$

By $UX=Y$

$$\begin{bmatrix}5 & -2 & 1\\ 0 & 19/5 & -32/5\\ 0 & 0 & 327/19\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 12/5\\ 46/19\end{bmatrix}$$

$$\Rightarrow \quad 5x-2y+z=4$$

$$\frac{19}{5}y-\frac{32}{5}z=\frac{12}{5}$$

$$\frac{327}{19}z=\frac{46}{19}$$

$$\therefore \quad z=\frac{46}{327}$$

$$\therefore \quad \frac{19}{5}y=\frac{12}{5}+\frac{32}{5}\left(\frac{46}{327}\right)$$

$$\Rightarrow \quad y=\frac{284}{327}$$

Hence, $\quad 5x=4+2y-z$

$$=4+\frac{568}{327}-\frac{46}{327}$$

$$\therefore \quad x=\frac{366}{327}$$

$$\therefore \quad x=\frac{366}{327}, y=\frac{284}{327}, z=\frac{46}{327}.$$

Ans.

Q. 17. Find the factors of the following matrix by using Cholesky decomposition method

$$U=\begin{bmatrix}1 & 2 & 3\\ 2 & 8 & 22\\ 3 & 22 & 82\end{bmatrix}.$$

(Bhopal 2011; Gwalior 16)

Solution : The given matrix is symmetric. Hence we will use here Cholesky method.

Let $\quad L=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}$

and $\quad L^T=\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$

Hence $\quad \begin{bmatrix}1 & 2 & 3\\ 2 & 8 & 22\\ 3 & 22 & 82\end{bmatrix}=\begin{bmatrix}l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}\begin{bmatrix}l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix}1 & 2 & 3\\ 2 & 8 & 22\\ 3 & 22 & 82\end{bmatrix}=\begin{bmatrix}l_{11}^2 & l_{11}l_{21} & l_{11}l_{31}\\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32}\\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2\end{bmatrix}$$

Comparing the corresponding elements both sides,

From first row $\quad l_{11}^2=1$ or $l_{11}=1$

$l_{11}l_{21}=2$ or $l_{21}=2$

$l_{11}l_{31}=3$ or $l_{31}=3$

From second row,

$$l_{21}^2+l_{22}^2=8\Rightarrow l_{22}=2$$

$$l_{21}l_{31}+l_{22}l_{32}=22\Rightarrow l_{32}=8$$

From third row,

$$l_{31}^2+l_{32}^2+l_{33}^2\Rightarrow l_{33}=3$$

$$L=\begin{bmatrix}1 & 0 & 0\\ 2 & 2 & 0\\ 3 & 8 & 3\end{bmatrix}.$$

Ans.

Q. 18. Solve by Gauss-Seidel method

$$10x_1-x_2=9$$

$$-x_1+5x_2=4.$$

(Indore 2011)

Solution : The given equation can be written as

$$x_1=\frac{1}{10}(9+x_2)$$

$$x_2 = \frac{1}{5}(4 + x_1)$$

In starting $x_2 = 0 \Rightarrow x_1 = \frac{9}{10} = 0{\cdot}9$ and $x_2 = \frac{1}{5}(4 + 0{\cdot}9) = 0{\cdot}98$

First Iteration $x_1^{(1)} = \frac{1}{10}(9 + 0{\cdot}98) = 0{\cdot}998$

Second Iteration $x_2^{(1)} = \frac{1}{5}(4 + 0{\cdot}998) = 0{\cdot}9996$

Third Iteration $x_1^{(2)} = \frac{1}{10}(9 + 0{\cdot}9996) = 0{\cdot}99996$

$$x_2^{(2)} = \frac{1}{5}(4 + 0{\cdot}99996) = 0{\cdot}999992$$

Fourth Iteration $x_1^{(3)} = \frac{1}{10}(9 + 0{\cdot}999992) = 0{\cdot}9999992$

$$x_2^{(3)} = \frac{1}{5}(4 + 0{\cdot}9999992) = 0{\cdot}99999984$$

Upto the three decimal places

$$x_1^{(2)} = x_1^{(3)} = 0{\cdot}999 \approx 1$$

and $x_2^{(2)} = x_2^{(3)} = 0{\cdot}999 \approx 1.$ **Ans.**

Q. 19. Solve the following system of equation by using Gauss-Seidel iterative method

$$2x_1 - x_2 + 0x_3 = 7; -x_1 + 2x_2 - x_3 = 1; 0x_1 - x_2 + 2x_3 = 1.$$

(Jabalpur 2005, 16; Gwalior 07, 16; Ujjain 12, 16; Indore 16)

Solution : Since $|2| > |-1| + |0|, |2| > |-1| + |-1|, |2| > |0| + |-1|$ *i.e.*, in each equation of the given system one of the coefficient is larger than the other and thereby satisfying the condition for Gauss-Seidal method, we write the given system of equations in the following form :

$$\left.\begin{aligned} x_1 &= \frac{1}{2}(7 + x_2) \\ x_2 &= \frac{1}{2}(1 + x_1 + x_3) \\ x_3 &= \frac{1}{2}(1 + x_2) \end{aligned}\right\} \quad \ldots(1)$$

We begin with $x_2 = x_3 = 0$ and using the most recent values of x_1, x_2, x_3 we get

First Iteration :

$$x_1^{(1)} = \frac{1}{2}[7 + x_2^{(0)}] = \frac{1}{2}[7 + 0] = 3{\cdot}5$$

$$x_2^{(1)} = \frac{1}{2}[1 + x_1^{(1)} + x_3^{(0)}] = \frac{1}{2}[1 + 3{\cdot}5 + 0] = 2{\cdot}25$$

$$x_3^{(1)} = \frac{1}{2}[1 + x_2^{(1)}] = \frac{1}{2}[1 + 2{\cdot}25] = 1{\cdot}625.$$

Second Iteration :

$$x_1^{(2)} = \frac{1}{2}[7 + x_2^{(1)}] = \frac{1}{2}[7 + 2{\cdot}25] = 4{\cdot}625$$

$$x_2^{(2)} = \frac{1}{2}[1 + x_1^{(2)} + x_3^{(1)}] = \frac{1}{2}[1 + 4{\cdot}625 + 1{\cdot}625] = 3{\cdot}625$$

$$x_3^{(2)} = \frac{1}{2}[1 + x_2^{(3)}] = \frac{1}{2}[1 + 3{\cdot}625] = 2{\cdot}3125.$$

Third Iteration :

$$x_1^{(3)} = \frac{1}{2}[7 + x_2^{(2)}] = \frac{1}{2}[7 + 3{\cdot}625] = 5{\cdot}3125$$

$$x_2^{(3)} = \frac{1}{2}[1 + x_1^{(3)} + x_3^{(2)}] = \frac{1}{2}[1 + 5{\cdot}3125 + 2{\cdot}3125] = 4{\cdot}3125$$

$$x_3^{(3)} = \frac{1}{2}[1 + x_2^{(3)}] = \frac{1}{2}[1 + 4{\cdot}3125] = 2{\cdot}65625.$$

Fourth Iteration :

$$x_1^{(4)} = \frac{1}{2}[7 + x_2^{(3)}] = \frac{1}{2}[7 + 4{\cdot}3125] = 5{\cdot}65625$$

$$x_2^{(4)} = \frac{1}{2}[1 + x_1^{(4)} + x_3^{(3)}] = \frac{1}{2}[1 + 5{\cdot}65625 + 2{\cdot}65625] = 4{\cdot}65625$$

$$x_3^{(4)} = \frac{1}{2}[1 + x_2^{(4)}] = \frac{1}{2}[1 + 4{\cdot}65625] = 2{\cdot}828125.$$

Fifth Iteration :

$$x_1^{(5)} = \frac{1}{2}[7 + x_2^{(4)}] = \frac{1}{2}[7 + 4{\cdot}65625] = 5{\cdot}828125$$

$$x_2^{(5)} = \frac{1}{2}[1 + x_1^{(5)} + x_3^{(4)}] = \frac{1}{2}[1 + 5{\cdot}828125 + 2{\cdot}828125] = 4{\cdot}828125$$

$$x_3^{(5)} = \frac{1}{2}[1 + x_2^{(5)}] = \frac{1}{2}[1 + 4{\cdot}828125] = 2{\cdot}9140625.$$

Sixth Iteration :

$$x_1^{(6)} = \frac{1}{2}[7 + x_2^{(5)}] = \frac{1}{2}[7 + 4{\cdot}828125] = 5{\cdot}9140625$$

$$x_2^{(6)} = \frac{1}{2}[1 + x_1^{(6)} + x_3^{(5)}] = \frac{1}{2}[1 + 5{\cdot}9140625 + 2{\cdot}9140625] = 4{\cdot}9140625$$

$$x_3^{(6)} = \frac{1}{2}[1 + x_2^{(6)}] = \frac{1}{2}[1 + 4{\cdot}9140625] = 2{\cdot}95703125.$$

Seventh Iteration :

$$x_1^{(7)} = \frac{1}{2}[7 + x_2^{(6)}] = \frac{1}{2}[7 + 4{\cdot}9140625] = 5{\cdot}95703125$$

$$x_2^{(7)} = \frac{1}{2}[1 + x_1^{(7)} + x_3^{(6)}] = \frac{1}{2}[1 + 5{\cdot}95703125 + 2{\cdot}95703125] = 4{\cdot}95703125$$

$$x_3^{(7)} = \frac{1}{2}[1 + x_2^{(7)}] = \frac{1}{2}[1 + 4{\cdot}95703125] = 2{\cdot}97851563.$$

Eighth Iteration :

$$x_1^{(8)} = \frac{1}{2}[7 + x_2^{(7)}] = \frac{1}{2}[7 + 4{\cdot}95703125] = 5{\cdot}97851563$$

$$x_2^{(8)} = \frac{1}{2}[1 + x_1^{(8)} + x_3^{(7)}] = \frac{1}{2}[1 + 5{\cdot}97851563 + 2{\cdot}97851563]$$

$$= 4{\cdot}97851563$$

$$x_3^{(8)} = \frac{1}{2}[1 + x_2^{(8)}] = \frac{1}{2}[1 + 4{\cdot}97851563] = 2{\cdot}98925782.$$

Nineth Iteration :

$$x_1^{(9)} = \frac{1}{2}[7 + x_2^{(8)}] = \frac{1}{2}[7 + 4{\cdot}97851563] = 5{\cdot}98925782$$

$$x_2^{(9)} = \frac{1}{2}[1 + x_1^{(8)} + x_3^{(7)}] = \frac{1}{2}[1 + 5{\cdot}98925782 + 2{\cdot}98925782]$$

$$= 4{\cdot}98925782$$

$$x_3^{(9)} = \frac{1}{2}[1 + x_2^{(9)}] = \frac{1}{2}[1 + 4{\cdot}98925782] = 2{\cdot}99462891.$$

Tenth Iteration :

$$x_1^{(10)} = \frac{1}{2}[7 + x_2^{(9)}] = \frac{1}{2}[7 + 4{\cdot}98925782] = 5{\cdot}99462891$$

$$x_2^{(10)} = \frac{1}{2}[1 + x_1^{(10)} + x_3^{(9)}] = \frac{1}{2}[1 + 5{\cdot}99462891 + 2{\cdot}99462891]$$

$$= 4{\cdot}98925782$$

$$x_3^{(10)} = \frac{1}{2}[1 + x_2^{(10)}] = \frac{1}{2}[1 + 4{\cdot}99462891] = 2{\cdot}99731446.$$

Eleventh Iteration :

$$x_1^{(11)} = \frac{1}{2}[7 + x_2^{(10)}] = \frac{1}{2}[7 + 4{\cdot}99462891] = 5{\cdot}99731446$$

$$x_2^{(11)} = \frac{1}{2}[1 + x_1^{(11)} + x_3^{(10)}] = \frac{1}{2}[1 + 5{\cdot}99731446 + 2{\cdot}99731446] = 4{\cdot}99731446$$

$$x_3^{(11)} = \frac{1}{2}[1 + x_2^{(11)}] = \frac{1}{2}[1 + 4{\cdot}99731446] = 2{\cdot}99865723.$$

Since the values in 10^{th} and 11^{th} iterations are practically same, we can stop the iteration process.

Hence, the required solution upto four decimal places is :

$x_1 = 5{\cdot}9973, x_2 = 4{\cdot}9973, x_3 = 2{\cdot}9987.$ **Ans.**

Q. 20. (a) Solve by Gauss-Seidel method

$20x + y - 2z = 17;\ 3x + 20y - z = -18;\ 2x - 3y + 20z = 25.$ **(Indore 2016)**

Solution : Since $|20| > |1| + |-2|, |20| > |3| + |-1|, |20| > |2| + |-3|$ *i.e.*, in each equation of the given system one of the coefficient is larger than the other and thereby satisfying the condition for Gauss-Seidal method, we write the given system of equations in the following form :

$$x = \frac{1}{20}(17 - y + 2z),\ y = \frac{1}{20}(-18 - 3x + z),\ z = \frac{1}{20}(25 - 2x + 3y) \qquad \ldots(1)$$

We begin with $y = z = 0$ and using the most recent values of x, y, z, we get

First Iteration :

$$x^{(1)} = \frac{1}{20}[17 - y^{(0)} + 2z^{(0)}] = \frac{1}{20}[17 - 0 + 0] = 0{\cdot}8500$$

$$y^{(1)} = \frac{1}{20}[-18 - 3x^{(1)} + z^{(0)}] = -1{\cdot}0275$$

$$z^{(1)} = \frac{1}{20}[25 - 2x^{(1)} + 3y^{(1)}] = 1{\cdot}0109$$

Second Iteration :

$$x^{(2)} = \frac{1}{20}[17 - y^{(1)} + 2z^{(1)}] = 1{\cdot}0025$$

$$y^{(2)} = \frac{1}{20}[-18 - 3x^{(2)} + z^{(1)}] = -0{\cdot}9998$$

$$z^{(2)} = \frac{1}{20}[25 - 2x^{(2)} + 3y^{(2)}] = 0{\cdot}9998$$

Third Iteration :

$$x^{(3)} = \frac{1}{20}[17 - y^{(2)} + 2z^{(2)}] = 1{\cdot}0000$$

$$y^{(3)} = \frac{1}{20}[-18 - 3x^{(3)} + z^{(2)}] = -1{\cdot}0000$$

$$z^{(3)} = \frac{1}{20}[25 - 2x^{(3)} + 3y^{(3)}] = 1{\cdot}0000$$

Since, the values in the 2^{nd} and 3^{rd} iterations being practically the same can stop the iteration process. Hence the required solution is :

$x = 1, y = -1, z = 1.$ **Ans.**

Q. 20. (b) Solve the following system of equation by using Gauss-Seidel iteration method.

$3x + 2y + 7z = 4;\ 2x + 3y + z = 5;\ 3x + 4y + z = 7.$ **(Indore 2022)**

Solution : Solve as Q. No. 20 (a).

Q. 20. (c) Solve by Gauss-seidel Iterative method :

$10x+2y+z=9;\ 2x+20y-2z=-44;\ -2x+3y+10z=22.$ **(Gwalior 2017)**

Solution : Solve as Q. No. 20(a).

Q. 21. Solve by Cholesky method :

$x+2y+3z=5,\ 2x+8y+22z=6,\ 3x+22y+82z=-10.$ **(Gwalior 2017; Ujjain 17)**

Solution : The given system of linear equation can be written in the form of matrix

$$AX=B$$

Where $$A=\begin{bmatrix}1&2&3\\2&8&22\\3&22&82\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}5\\6\\-10\end{bmatrix}$$

Let $L=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}$ then the transpose of it is

$$L^T=\begin{bmatrix}l_{11}&l_{21}&l_{31}\\0&l_{22}&l_{32}\\0&0&l_{33}\end{bmatrix}.$$

We write

$$A=LL^T$$

$$\begin{bmatrix}1&2&3\\2&8&22\\3&22&82\end{bmatrix}=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}\begin{bmatrix}l_{11}&l_{21}&l_{31}\\0&l_{22}&l_{32}\\0&0&l_{33}\end{bmatrix}$$

$$=\begin{bmatrix}l_{11}^2&l_{11}l_{21}&l_{11}l_{31}\\l_{21}l_{11}&l_{21}^2+l_{22}^2&l_{21}l_{31}+l_{22}l_{32}\\l_{31}l_{11}&l_{31}l_{21}+l_{32}l_{22}&l_{31}^2+l_{32}^2+l_{33}^2\end{bmatrix}.$$

Comparison the element of the both sides, we get

First Row : $l_{11}^2=1$ or $l_{11}=1$,

$l_{11}l_{21}=2$ or $l_{21}=2$,

$l_{11}l_{31}=3$ or $l_{31}=3$,

Second Row : $l_{21}^2+l_{22}^2=8$ or $l_{22}=2$,

$l_{21}l_{31}+l_{22}l_{32}=22$ or $l_{32}=8$,

Third Row : $l_{31}^2+l_{32}^2+l_{33}^2=82$ or $l_{33}=3$.

Hence, we get

$$A=LL^T, \text{ where } L=\begin{bmatrix}1&0&0\\2&2&0\\3&8&3\end{bmatrix}.$$

The system of equation can be written as

$$LL^TX=B$$

or $LY=B$ or $L^TX=Y=\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$, let

From $LY=B$, we get

$$\begin{bmatrix}1&0&0\\2&2&0\\3&8&3\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}5\\6\\-10\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}y_1\\2y_1+2y_2\\3y_1+8y_2+3y_3\end{bmatrix}=\begin{bmatrix}5\\6\\-10\end{bmatrix}$$

$$\Rightarrow \quad y_1=5,\ 2y_1+2y_2=6,\ 3y_1+8y_2+3y_3=-10$$

$$\Rightarrow \quad y_1=5,\ y_2=-2,\ y_3=-3.$$

$$\therefore \quad Y=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\-2\\-3\end{bmatrix}.$$

From $L^TX=Y$, we get

$$\begin{bmatrix}1&2&3\\0&2&8\\0&0&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5\\-2\\-3\end{bmatrix}$$

$$\Rightarrow \quad x+2y+3z=5,\ 2y+8z=-2,\ 3z=-3$$

$$\Rightarrow \quad x=2,\ y=3,\ z=-1.$$

Hence the required solution is $x=2,\ y=3,\ z=-1$. **Ans.**

Q. 22. What do you understand by Iterative method ? Explain Jacobi Iterative method and Gauss-Seidel Iterative method. **(Jabalpur 2012)**

Ans. Simultaneous linear equations are solved by following two methods :

(a) Direct method which are studied earlier

(b) Indirect method or Iterative method.

In Iterative method we start with an approximate solution and then we derive a sequence of closer approximations. The process of computation is repeated so as to obtain the desired accuracy.

(i) Jacobi Iterative method : Let us consider the system of simultaneous linear equations as : **(Bhopal 2011)**

$$\left.\begin{aligned}a_{11}x_1+a_{12}x_2+a_{13}x_3&=b_1\\a_{21}x_1+a_{22}x_2+a_{23}x_3&=b_2\\a_{31}x_1+a_{32}x_2+a_{33}x_3&=b_3\end{aligned}\right\} \quad \ldots(1)$$

Let the diagonal coefficients be large as compared to other coefficients and do not vanish. In case any of these vanish then rearrange the equation to satisfy this condition.

Now write the system (1) as below :

$$\left.\begin{aligned} x_1 &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2 - \frac{a_{13}}{a_{11}} x_3 \\ x_2 &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - \frac{a_{23}}{a_{22}} x_3 \\ x_3 &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1 - \frac{a_{32}}{a_{33}} x_2 \end{aligned}\right\} \quad \ldots (2)$$

Now we start with initial approximation each equal to zero to the unknown x_1, x_2, x_3. Putting these values on the R.H.S. of eqn. (2)

$$x_1 = \frac{b_1}{a_{11}}, \; x_2 = \frac{b_2}{a_{22}}, \; x_3 = \frac{b_3}{a_{33}}$$

Let these first approximation be denoted by $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ respectively. On putting these values in the R.H.S. of eqn. (2), a system of second approximation is obtained and is given by :

$$x_1^{(2)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \frac{a_{13}}{a_{11}} x_3^{(1)}$$

$$x_2^{(2)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(1)}$$

$$x_3^{(2)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \frac{a_{33}}{a_{33}} x_2^{(1)}$$

Above process is repeated untill two successive approximations are same.

(ii) Gauss-Seidel iterative method : Gauss-Seidel method is modified form of Jacobi method. As above in Jacobi's iteration method, we take first approximation as $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ and put in the right hand side of first eqn. (2) and suppose the result is $x_1^{(2)}$. Now we put $x_1 = x_1^{(2)}, x_3 = x_3^{(1)}$ in the R.H.S. of second eqn. (2) and suppose the result is $x_2^{(2)}$. Now put $x_1 = x_1^{(2)}, x_2 = x_2^{(2)}$ in the R.H.S. of third eqn. (2) and suppose the result in $x_2^{(2)}$. Note that as soon as a new approximation for an unknown is obtained it is immediately used in the next step. This entire process is repeated till the values of unknown x_1, x_2, x_3..... are found to desired accuracy.

Note : The convergence in Gauss-Seidel method is more than fast that in Jacobi's method. Since, in Gauss-Seidel method the most current approximation is used.

Q. 23. Solve by using Jacobi-Iteration method

$$10x_1 + 2x_2 + x_3 = 9$$
$$2x_1 + 20x_2 - 2x_3 = -44$$
$$-2x_1 + 3x_2 + 10x_3 = 22.$$

(Indore 2012; Bhopal 16)

Solution :

$\because$ $|a_{11}| > |a_{12}| + |a_{13}|$

$|a_{22}| > |a_{21}| > |a_{23}|$

$|a_{33}| > |a_{31}| > |a_{32}|$

The system can be written as

$$x_1^{(k+1)} = \frac{1}{10}(9 - 2x_2^{(k)} - x_3^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{20}(-44 - 2x_1^{(k)} + 2x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{10}(22 + 2x_1^{(k)} - 3x_2^{(k)})$$

For initial approximation let $x^{(0)} = [0, 0, 0]$

Taking $k = 0$ the next approximation is

$$x_1^{(1)} = \frac{9}{10} = 0{\cdot}9$$

$$x_2^{(1)} = \frac{-44}{20} = -2{\cdot}2$$

$$x_3^{(1)} = \frac{22}{10} = 2{\cdot}2$$

Taking $k = 1,2,3,$ ..... the successive approximations are

$$x_1^{(2)} = \frac{1}{10} \times [9 - 2(-2{\cdot}2) - 2{\cdot}2] = 1{\cdot}12$$

$$x_2^{(2)} = \frac{1}{20} \times [-44 - 2(0{\cdot}9) + 2(2{\cdot}2)] = -2{\cdot}07$$

$$x_3^{(2)} = \frac{1}{10} \times [22 + 2(0{\cdot}9) - 3(-2{\cdot}2)] = 3{\cdot}04$$

$$x_1^{(3)} = \frac{1}{10} \times [9 - 2(-2{\cdot}07) - 3{\cdot}04] = 1{\cdot}01$$

$$x_2^{(3)} = \frac{1}{20} \times [-44 - 2(1{\cdot}12) + 2(3{\cdot}04)] = -2{\cdot}008$$

$$x_3^{(3)} = \frac{1}{10} \times [22 + 2(1{\cdot}12) - 3(-2{\cdot}07)] = -3{\cdot}045$$

$$x_1^{(4)} = \frac{1}{10} \times [9 - 2(-2{\cdot}008) - 3{\cdot}045] = 0{\cdot}997$$

$$x_2^{(4)} = \frac{1}{20} \times [-44 - 2(1{\cdot}01) + 2(3{\cdot}045)] = -1{\cdot}997$$

$$x_3^{(4)} = \frac{1}{10} \times [22 + 2(1{\cdot}01) - 3(-2{\cdot}008)] = 3{\cdot}004$$

$$x_1^{(5)} = \frac{1}{10} \times [9 - 2(-1{\cdot}997) - 3{\cdot}004] = 0{\cdot}999$$

$$x_2^{(5)} = \frac{1}{20} \times [-44 - 2(0{\cdot}997) + 2(3{\cdot}004)] = -1{\cdot}999$$

$$x_3^{(5)} \times \frac{1}{10} = [22 + 2(0{\cdot}997) - 3(-1{\cdot}997)] = 2{\cdot}998$$

Approximation solution set is $\{1, -2, 3\}$ (till $2d$). **Ans.**

Q. 24. Use Jacobi's method and solve the system of equations :

$$6x + 3y + 12z = 35$$
$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33.$$

(Indore 2017)

Solution : Since $|12| > |6| + |3|$, $|8| > |-3| + |2|$ and $|11| > |4| + |-1|$

So we use Jacobi's method for the given system of equations.

Hence we write x, y, z as follows :

$$x = \frac{1}{8}(20 + 3y - 2z), y = \frac{1}{11}(33 - 4x + z), z = \frac{1}{12}(35 - 6x - 3y) \quad ...(1)$$

Let the initial values be $x = 0, y = 0, z = 0$. Then, we get

First Iteration :

$$x^{(1)} = \frac{1}{8}[20 + 3y^{(0)} - 2z^{(0)}] = \frac{1}{8}[20 + 0 - 0] = 2{\cdot}5$$

$$y^{(1)} = \frac{1}{11}[33 - 4x^{(0)} - z^{(0)}] = \frac{1}{11}[33 - 0 + 0] = 3{\cdot}0$$

$$z^{(1)} = \frac{1}{12}[35 - 6x^{(0)} - 3y^{(0)}] = \frac{1}{12}[35 - 0 - 0] = 2{\cdot}916666$$

Second Iteration :

$$x^{(2)} = \frac{1}{8}[20 + 3y^{(1)} - 2z^{(1)}] = \frac{1}{8}[20 + 3(3{\cdot}0) - 2(2{\cdot}916666)] = 2{\cdot}895833$$

$$y^{(2)} = \frac{1}{11}[33 - 4x^{(1)} + z^{(1)}] = \frac{1}{11}[33 - 4(2{\cdot}5) + 2{\cdot}916666] = 2{\cdot}356060$$

$$z^{(2)} = \frac{1}{12}[35 - 6x^{(1)} - 3y^{(1)}] = \frac{1}{12}[35 - 6(2{\cdot}5) - 3(3{\cdot}0)] = 0{\cdot}916666$$

Third Iteration :

$$x^{(3)} = \frac{1}{8}[20 + 3y^{(2)} - 2z^{(2)}] = \frac{1}{8}[20 + 3(2{\cdot}356060) - 2(0{\cdot}916666)] = 3{\cdot}154356$$

$$y^{(3)} = \frac{1}{11}[33 - 4x^{(2)} + z^{(2)}] = \frac{1}{11}[33 - 4(2{\cdot}895833) + 0{\cdot}916666] = 2{\cdot}030303$$

$$z^{(3)} = \frac{1}{12}[35 - 6x^{(2)} - 3y^{(2)}] = \frac{1}{12}[35 - 6(2{\cdot}895833) - 3(2{\cdot}356060)] = 0{\cdot}879735$$

Fourth Iteration :

$$x^{(4)} = \frac{1}{8}[20 + 3(2{\cdot}030303) - 2(0{\cdot}879735)] = 3{\cdot}041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4(3{\cdot}154356) + 0{\cdot}879735] = 1{\cdot}932937$$

$$z^{(4)} = \frac{1}{12}[35 - 6(3{\cdot}154356) - 3(2{\cdot}030303)] = 0{\cdot}831913$$

Fifth Iteration :

$$x^{(5)} = \frac{1}{8}[20 + 3(1{\cdot}932937) - 2(0{\cdot}831913)] = 3{\cdot}016873$$

$$y^{(5)} = \frac{1}{11}[33 - 4(3{\cdot}041430) + 0{\cdot}831913] = 1{\cdot}969654$$

$$z^{(5)} = \frac{1}{12}[35 - 6(3{\cdot}041430) - 3(1{\cdot}932937)] = 0{\cdot}912717$$

Sixth Iteration :

$$x^{(6)} = \frac{1}{8}[20 + 3(1{\cdot}969654) - 2(0{\cdot}912717)] = 3{\cdot}010441$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3{\cdot}016873) + 0{\cdot}912717] = 1{\cdot}985930$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3{\cdot}016873) - 3(1{\cdot}969654)] = 0{\cdot}915817$$

Seventh Iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3(1{\cdot}985930) - 2(0{\cdot}915817)] = 3{\cdot}015770$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3{\cdot}010441) + (0{\cdot}915817)] = 1{\cdot}988550$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3{\cdot}010441) - 3(1{\cdot}985930)] = 0{\cdot}914964$$

Eighth Iteration :

$$x^{(8)} = \frac{1}{8}[20 + 3(1{\cdot}988550) - 2(0{\cdot}914964)] = 3{\cdot}016946$$

$$y^{(8)} = \frac{1}{11}[33 - 4(3{\cdot}015770) + (0{\cdot}914964)] = 1{\cdot}986535$$

$$z^{(8)} = \frac{1}{12}[35 - 6(3{\cdot}015770) - 3(1{\cdot}988550)] = 0{\cdot}911644$$

Ninth Iteration :

$$x^{(9)} = \frac{1}{8}[20 + 3(1{\cdot}986535) - 2(0{\cdot}911644)] = 3{\cdot}017040$$

$$y^{(9)} = \frac{1}{11}[33 - 4(3{\cdot}016946) + (0{\cdot}911644)] = 1{\cdot}985805$$

$$z^{(9)} = \frac{1}{12}[35 - 6(3{\cdot}016946) - 3(1{\cdot}986535)] = 0{\cdot}911560$$

Tenth Iteration :

$$x^{(10)} = \frac{1}{8}[20 + 3(1{\cdot}985805) - 2(0{\cdot}911560)] = 3{\cdot}016787$$

$$y^{(10)} = \frac{1}{11}[33 - 4(3{\cdot}017040) + (0{\cdot}911560)] = 1{\cdot}985764$$

$$z^{(10)} = \frac{1}{12}[35 - 6(3{\cdot}017040) - 3(1{\cdot}985805)] = 0{\cdot}911695$$

Since ninth and tenth iterations are practically same correct to three decimal places, hence the required solution is

$x = 3{\cdot}016, y = 1{\cdot}985, z = 0{\cdot}911.$ **Ans.**

Q. 25. Solve by Jacobi Iteration method

$$20x_1 + 2x_2 + 6x_3 = 28$$
$$x_1 + 20x_2 + 9x_3 = -23$$
$$2x_1 - 7x_2 - 20x_3 = -57.$$

(Indore 2011; Jabalpur 16)

Solution : Solve as Q. No. 24.

Q. 26. Solve the following by Jacobi method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110.$$

(Gwalior 2016; Indore 16; Jabalpur 17)

Solution : (a) Jacobi method : Solving each equation of the given system for the unknown with the largest coefficients in terms of the remaining unknowns.

$$\left.\begin{aligned} x &= \frac{1}{27}(85 - 6y + z) \\ y &= \frac{1}{15}(72 - 6x - 2z) \\ z &= \frac{1}{54}(110 - x - y) \end{aligned}\right\} \quad ...(1)$$

Starting with initial approximation $x = 0, y = 0, z = 0$.

First approximation is :

$$x^{(1)} = \frac{85}{27} = 3{\cdot}184,\ y^{(1)} = \frac{72}{15} = 4{\cdot}8,\ z^{(1)} = \frac{110}{54} = 2{\cdot}037$$

Now second approximation :

$$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)}) = \frac{1}{27}(85 - 28{\cdot}8 + 2{\cdot}037) = 2{\cdot}157$$

$$y^{(2)} = \frac{1}{15}(72 - 6x^{(1)} + 2z^{(1)}) = \frac{1}{15}(72 - 18{\cdot}888 - 4{\cdot}074) = 3{\cdot}269$$

$$z^{(2)} = \frac{1}{54}(110 - x^{(1)} - y^{(1)}) = \frac{1}{54}(110 - 3{\cdot}148 - 4{\cdot}8) = 1{\cdot}890$$

Now third approximation :

$$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)}) = 2{\cdot}492$$

$$y^{(3)} = \frac{1}{15}(72 - 6x^{(2)} - 2z^{(2)}) = 3{\cdot}685$$

$$z^{(3)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)}) = 1{\cdot}937$$

Similarly,

$$x^{(4)} = \frac{1}{27}(85 - 6y^{(3)} + z^{(3)}) = 2{\cdot}401$$

$$y^{(4)} = \frac{1}{15}(72 - 6x^{(3)} - 2z^{(3)}) = 3{\cdot}545$$

$$z^{(4)} = \frac{1}{54}(110 - x^{(3)} - y^{(3)}) = 1{\cdot}923$$

$$x^{(5)} = \frac{1}{27}(85 - 6y^{(4)} + z^{(4)}) = 2{\cdot}432$$

$$y^{(5)} = \frac{1}{15}(72 - 6x^{(4)} - 2z^{(4)}) = 3{\cdot}583$$

$$z^{(5)} = \frac{1}{54}(110 - x^{(4)} - y^{(4)}) = 1{\cdot}927$$

$$x^{(6)} = 2{\cdot}423,\ y^{(6)} = 3{\cdot}570,\ z^{(6)} = 1{\cdot}926$$

$$x^{(7)} = 2{\cdot}426,\ y^{(7)} = 3{\cdot}574,\ z^{(7)} = 1{\cdot}926$$

Sixth and seventh iterations give same values, therefore we can stop.

Hence, set of solution (2·426, 3·574, 1·926). **Ans.**

Q. 27. Explain Choleski's decomposition method and Solve the following system of equations by using this method :

$$4x_1 + 2x_2 - 2x_3 = 4$$
$$2x_1 + 10x_2 + 2x_3 = 14$$
$$-2x_1 + 2x_2 + 3x_3 = 3.$$

(Indore 2011; Gwalior 11, 13; Ujjain 13; Jabalpur 17)

Solution : In special condition symmetrical and positive definite matrices can be constructed as alternative triangular factorization, it saves time.

In this method $AX = B$ can be represented as $LL^TX = B$

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$ then

$L = \begin{bmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \cdots & \cdots & & \cdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}$ and $L^T = \begin{bmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ & & & \cdots \\ 0 & & & l_{nn} \end{bmatrix}$.

Then $a_{11} = l_{11}^2$, $a_{12} = l_{11}l_{21}$, ..., $a_{in} = l_{11}l_{n1}$,

$a_{21} = l_{21}l_{11}$, $a_{22} = l_{21}^2 + l_{22}^2$, $a_{23} = l_{21}l_{31} + l_{22}l_{32}$, ...

$a_{2n} = a_{21}l_{n1} + l_{22}l_{n2}$,

... ...

$a_{nn} = l_{n1}^2 + l_{n2}^2 + \ldots\ldots + l_{nn}^2$

Elements of L can be obtained. It is possible that some elements are totally imaginary, but it does not create any problem. Hence, we assume that all elements are real.

If L is real then $A = LL^T$ is positive definite matrix. Because $X^TAX = X^TLL^TX = (X^TL)(L^TX) = Z^TZ$. When $Z = L^TX$, but Z is real hence $Z^TZ > 0$, in other hand if A is positive definite then L is real system $AX = B$ can be solve in two steps,

$LL^TX = B \Rightarrow LZ = B$ and $L^TX = Z$,

This method is also known as square root method.

Coefficient matrix in given system,

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix},$$

(from Cholaski's decomposition method)

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Step 1 : On comparing elements of first column,

$4 = l_{11}^2 \Rightarrow l_{11} = 2$ or -2, (Hence, matrix L is not unique)

$2 = l_{21}l_{11} \Rightarrow l_{21} = 1$, (if $l_{11} = 2$)

$-2 = l_{31}l_{11} \Rightarrow l_{31} = -1$

Step 2 : On comparing elements of second column,

$10 = l_{21}^2 + l_{22}^2 \Rightarrow l_{22} = \sqrt{(10 - l_{21}^2)} = \sqrt{10-1} = 3$, (on taking + ve sign)

$2 = l_{31}l_{21} + l_{32}l_{22} \Rightarrow l_{32} = \dfrac{2-(-1)(1)}{3} = \dfrac{2+1}{3} = 1.$

Step 3 : On comparing elements of third column,

$3 = l_{31}^2 + l_{32}^2 + l_{33}^2 \Rightarrow l_{33} = \sqrt{1} = 1.$

Hence, $\begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

or $A = LL^T$

Now, $AX = B \Rightarrow LL^TX = B \Rightarrow LC = B$ where $L^TX = C$

$\therefore \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 3 \end{bmatrix} \Rightarrow c_1 = 2, c_2 = 4, c_3 = 1$

and from $L^TX = C$, $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$

$\Rightarrow \quad x_3 = 1, x_2 = 1, x_1 = 1.$

Solution set is (1, 1, 1). **Ans.**

Q. 28. Apply Gauss-Jordan method to solve the equations :

$$x + y + z = 9;\ 2x - 3y + 4z = 13;\ 3x + 4y + 5z = 40.$$ **(Indore 2017)**

Solution : We have

$$x + y + z = 9 \quad \ldots(1)$$

$$2x - 3y + 4z = 13 \quad \ldots(2)$$

$$3x + 4y + 5z = 40 \quad \ldots(3)$$

For eliminating of x,

From eqn. (2) – 2 × eqn. (1),

$$-5y + 2z = -5 \quad \ldots(4)$$

and from eqn. (3) – 3 × eqn. (1),

$$y + 2z = 13 \quad \ldots(5)$$

For eliminating of y,

From eqn. (4) + 5 × eqn. (1),

$$5x + 7z = 40 \quad \ldots(6)$$

and from eqn. (4) + 5 × eqn. (5),

$$12z = 60$$

$$\Rightarrow \quad z = 5 \quad \ldots(7)$$

For eliminating of z,

From eqn. (4) – 2 × eqn. (7),

$$-5y = -15$$

$$\Rightarrow \quad y = 3$$

and from eqn. (6) – 7 × eqn. (7),

$$5x = 5$$

$$\Rightarrow \quad x = 1$$

Hence required solution is :

$$x = 1, y = 3, z = 5.$$ **Ans.**

Q. 29. Apply Gauss-Jordan method and solve the system of equations :

$10x + y + z = 12;\ 2x + 10y + z = 13;\ x + y + 5z = 7.$

(Jabalpur 2022; Indore 22)

Solution : The given system in matrix form is $AX = B$

$\therefore$ Augmented matrix, $[A:B] = \begin{bmatrix} 10 & 1 & 1 & : & 12 \\ 2 & 10 & 1 & : & 13 \\ 1 & 1 & 5 & : & 7 \end{bmatrix}$

Operating R_{13}, we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 2 & 10 & 1 & : & 13 \\ 10 & 1 & 1 & : & 12 \end{bmatrix}$$

Taking $a_{11} = 1$ as pivot operating $R_{21}(-2), R_{31}(-10)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & 8 & -9 & : & -1 \\ 0 & -9 & -49 & : & -58 \end{bmatrix}$$

Operating $R_{21}(1)$ we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 5 & : & 7 \\ 0 & -1 & -58 & : & -59 \\ 0 & -9 & -49 & : & -58 \end{bmatrix}$$

Taking $b_{22} = -1$ as pivot, operating $R_{12}(1), R_{32}(-9)$, we get

$$[A:B] = \begin{bmatrix} 1 & 0 & -53 & : & -52 \\ 0 & -1 & -58 & : & -59 \\ 0 & 0 & 473 & : & 473 \end{bmatrix}$$

Operating $R_3\left(\frac{1}{473}\right)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 0 & -53 & : & -52 \\ 0 & -1 & -58 & : & -59 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

Taking $c_{33} = 1$ as pivot, operating $R_{13}(53), R_{23}(58)$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 \\ 0 & -1 & 0 & : & -1 \\ 0 & 0 & 1 & : & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad x = 1,\ y = 1,\ z = 1.$$

Ans.

Q. 30. Solve by Gauss-Jordan method

$x + 2y + z = 8;\ 2x + 3y + 4z = 20;\ 4x + 3y + 2z = 16.$ **(Chhindwara 2022)**

Solution : Solve as Q. No. 29.

Objective Type Questions

1. The form of LU decomposition method where L is a lower triangular matrix : **(Ujjain 2022)**

(a) $\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) None of these.

2. The first iterate of system of equations by Gauss-Seidel method :

$5x_1 + 2x_2 - x_3 = 6,\ x_1 + 6x_2 - 3x_3 = 4,\ 2x_1 + x_2 + 4x_3 = 7.$ **(Indore 2014)**

(a) (1·2, 0·467, 1·033) (b) (1·2, 0, 0)

(c) (1·2, 0·467, 0) (d) None of these.

3. By Jacobi iterative method, if the initial solution of the following system of equation is (0, 0, 0) then the first iterate will be :

$$10x - 5y - 2z = 3$$
$$4x - 10y + 3z = -3$$
$$x + 6y + 10z = -3.$$

(a) (−0·3, 0·3, 0·3) (b) (0·3, −0·3, 0·3)

(c) (0·3, 0·3, −0·3) (d) (0·3, −0·3, −0·3).

4. By Jacobi iterative method if the initial solution of the following system of equation is (2, 3, 0) then the first iterate will be :

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1.$$

(a) (2·6, 2·3, −1·1) (b) (2·6, 2, −1·2)

(c) (2, 2·6, −1·2) (d) (2, 2·6, −1·4).

5. Jacobi iterative method is :

(a) $x^{(k+1)} = D^{-1}[b - (L+U)x^{(k)}]$ (b) $x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b$

(c) $x^{(k+1)} = D[b + (L+U)x^{(k)}]$ (d) $x^{(k+1)} = (D+L)^{-1}Ux^{(k)} - (D+L)^{-1}b.$

6. **Gauss-Seidel method is :**

(a) $x^{(k+1)} = -D^{-1}(Lx^{(k)} + Ux^{(k)}) + D^{-1}b$

(b) $x^{(k+1)} = D^{-1}(Lx^{(k)} + Ux^{(k)}) - D^{-1}b$

(c) $x^{(k+1)} = -D^{-1}(Lx^{(k+1)} + Ux^{(k)}) + D^{-1}b$

(d) $x^{(k+1)} = -D^{-1}(Lx^{(k)} + Ux^{(k+1)}) + D^{-1}b.$

7. **The first iterate of the following system of equations by Gauss-Seidel method :**

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1.$$

(a) (2·5, 3·5, 1·25) (b) (3·5, 2·5, 1·625)
(c) (1·5, 2·5, 1·25) (d) (3·5, 2·25, 1·625).

8. **The first iterate of the following system of equations by Gauss-Seidel method :**

$$5x_1 + 2x_2 - x_3 = 6$$
$$x_1 + 6x_2 - 3x_3 = 4$$
$$2x_1 + x_2 + 4x_3 = 7.$$

(a) (1·2, 0·467, 1·033) (b) (1·2, 0, 0)
(c) (1·2, 0·467, 0) (d) None of these.

9. **Gauss Elimination method is :**
(Bhopal 2014, 15; Indore 16; Jabalpur 16; Chhatarpur 22)

(a) Direct method (b) Indirect method
(c) Two way method (d) Three way method.

10. **In LU decomposition method where U is a upper triangular matrix is form of :**

(a) $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ (b) $\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) None of these.

11. **Gauss-Seidel iterative method uses :**

(a) Precalculated values from present iteration

(b) $x_i = (0)\ \forall\ i$

(c) $x_i \neq (0)$ for all i

(d) None of these.

12. **The solution of $2x - y + 3z = 9, x + y + z = 6, x - y + z = 2$ by using Gauss-Elimination method :**

(a) $x = 1, y = 2, z = 3$ (b) $x = 2, y = 2, z = 3$
(c) $x = 1, y = 1, z = 1$ (d) None of these.

Ans. 1. (b), 2. (c), 3. (c), 4. (b), 5. (a), 6. (d), 7. (d), 8. (a), 9. (a), 10. (a), 11. (a), 12. (a).

UNIT 4

Short and Long Answer Type Questions

Q. 1. (a) Write the name of any three methods for solving ordinary differential equation. **(Ujjain 2016)**

Ans. The name of methods for solving ordinary differential equation are :
(i) Taylor's method
(ii) Picard's method
(iii) Euler's method.

Q. 1. (b) Explain Picard's method for numerical solution of ordinary differential equation. **(Rewa 2017)**

Solution : To solve $\frac{dy}{dx} = f(x, y)$ subject to $y(x_0) = y_0$

Now $$\frac{dy}{dx} = f(x, y) \quad ...(1)$$

$\therefore$ $$dy = f(x, y)dx$$

Integrating,

$$y = \int^x f(x, y)dx + c \quad ...(2)$$

Setting $x = x_0$ on the R.H.S. after integration and $y = y_0$ on the L.H.S., we have

$$y_0 = \int^{x_0} f(x, y)dx + c \quad ...(3)$$

From eqns. (2) and (3), we have

$$y - y_0 = \int_{x_0}^{x} f(x, y)dx \quad ...(4)$$

$\therefore$ $$y = y_0 + \int_{x_0}^{x} f(x, y)dx \quad ...(5)$$

In equation (4) the R.H.S. integrand $f(x, y)$ involves y also. This type of equation is called an integral equation. As the integrations is not possible as it is, we will solve it by successive approximation.

Substitute the initial values of y namely y_0 in the integrand $f(x, y)$ in place of y and then integrate the R.H.S. to get an approximate value of y on the L.H.S.

i.e., $$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

Since $f(x, y_0)$ is a function of x also, it is possible to integrate it w.r.t. x.

After getting the first approximation $y^{(1)}$ for y use this value $y^{(1)}$ in the place y in $f(x, y)$ and then integrate to get the second approximation of y namely $y^{(2)}$.

i.e., $$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

Proceeding in this way, we get the n^{th} approximate value of y as

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

The above equation gives the general iterative formula for y. It is called Picard's iteration formula.

Q. 1. (c) Explain Euler's modified method for solving the ordinary differential equation.

Solution : Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0 \qquad ...(1)$$

Integrating with respect to between the limits x_n and x_{n+1}, we obtain

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y(x))\, dx \qquad ...(2)$$

The integral on the right hand side of (2) is approximated by the trapezoidal rule, which produces

$$y_{n+1} - y_n = \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})] \qquad ...(3)$$

There is however, an important difference between the Euler's formula and (3). In the latter case the unknown y_{n+1} is an argument of the function which needs to be solved at each step. The most convenient way of doing this is to use the fixed point interation, viz.

$$y_{n+1}^{(m+1)} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(m)})] \qquad ...(4)$$

A starting value $y_{n+1}^{(0)}$, has to be supplied, when may be calculated from Euler's formula as

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n) \qquad ...(5)$$

To compute y_{n+1} from a known value of y_n, we first calculate $y_{n+1}^{(0)}$ using (5) and then iterate with (4) using this as starting value. The iteration may be stopped either after a five number of iteration or when the quantity $|y_{n+1}^{(m+1)} - y_{n+1}^{(m)}|$ is less than a prescribed tolerance limit.

Used in this way, Euler's formula is called a predictor and the trapezoidal formula corrector.

Although, Euler's method is generally less accurate than the trapezoidal method. It provides a reasonable approximation and so only a few iterations should be needed to the convergence (4).

Q. 2. Use Picard's method the approximate y when $x = 0{\cdot}1$, $x = 0{\cdot}2$, given that $y = 0, x = 0, \frac{dy}{dx} = x + y$. **(Ujjain 2011, 14; Jabalpur 11, 12; Rewa 14; Sagar 14; Indore 15; Chhatarpur 22)**

Solution : We have $\frac{dy}{dx} = f(x, y) = x + y, x_0 = 0, y_0 = 0$

Now first approx,

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx = 0 + \int_0^x x\, dx = \frac{x^2}{2}$$

Second approx,

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx = 0 + \int_0^x \left\{x + \frac{x^2}{2}\right\} dx = \frac{x^3}{6} + \frac{x^2}{2}$$

Third approx,

$$y^{(3)} = y_0 + \int_0^x \left\{x + \left(\frac{x^3}{6} + \frac{x^2}{2}\right)\right\} dx = \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2}$$

When $x = 0{\cdot}1$, we have

$$y^{(1)} = \frac{(0{\cdot}1)^2}{2} = 0{\cdot}005$$

$$y^{(2)} = \frac{(0{\cdot}1)^3}{6} + \frac{(0{\cdot}1)^2}{2} = 0{\cdot}005167$$

$$y^{(3)} = \frac{(0{\cdot}1)^4}{24} + \frac{(0{\cdot}1)^3}{6} + \frac{(0{\cdot}1)^2}{2} = 0{\cdot}00517$$

When $x = 0{\cdot}2$ we have

$$y^{(1)} = \frac{(0{\cdot}2)^2}{2} = 0{\cdot}02$$

$$y^{(2)} = \frac{(0{\cdot}2)^3}{6} + \frac{(0{\cdot}2)^2}{2} = 0{\cdot}02133$$

$$y^{(3)} = \frac{(0{\cdot}2)^4}{24} + \frac{(0{\cdot}2)^3}{6} + \frac{(0{\cdot}2)^3}{2} = 0{\cdot}02139$$

We can continue further to get the better approx. **Ans.**

Q. 3. Solve $\frac{dy}{dx} = 2 - \frac{y}{x}$, $y(1) = 2$, for y by Picard's method upto the third approximation. **(Rewa 2011)**

Solution : Given $\frac{dy}{dx} = 2 - \frac{y}{x} = f(x, y)$ and $x_0 = 1, y_0 = 2$

First Iteration :

$$y_1 = y_0 + \int_{x_0}^{x} (x, y_0)\, dx$$

$$= 2 + \int_1^x \left(2 - \frac{y_0}{x}\right) dx$$

$$= 2 + \int_1^x \left(2 - \frac{2}{x}\right) dx$$

$$= 2 + (2x - 2\log x)_1^x = 2 + 2x - 2\log x - 2 + 0$$

$$= 2x - 2\log x$$

Second Iteration :

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$= 2 + \int_1^x \left(2 - \frac{y_1}{x}\right) dx$$

$$= 2 + \int_1^x \left(2 - \frac{2x - 2\log x}{x}\right) dx$$

$$= 2 + \int_1^x 2 \cdot \frac{1}{x} \log x \, dx$$

$$= 2 + 2\left[\frac{(\log x)^2}{2}\right]_1^x = 2 + (\log x)^2$$

Third Iteration :

$$y_3 = y_0 + \int_1^x f(x, y_2)\, dx$$

$$= 2 + \int_1^x \left(2 - \frac{y_2}{x}\right) dx$$

$$= 2 + \int_1^x 2 - \left(\frac{2 + (\log x)^2}{x}\right) dx$$

$$= 2 + \int_1^x \left\{2 - \frac{2}{x} - \frac{1}{x}(\log x)^2\right\} dx$$

$$= 2 + \left[2x - 2\log x - \frac{(\log x)^3}{3}\right]_1^x$$

$$= 2 + 2x - 2\log x - \frac{(\log x)^3}{3} - 2 + 2 \times 0 + \frac{0}{3}, \qquad (\because \log 1 = 0)$$

$$= 2x - 2\log x - \frac{(\log x)^3}{3}.$$ **Ans.**

Q. 4. Find the value of y for the constant value of x with the given condition of the given equation :

(a) $\frac{dy}{dx} = 1 + xy$ **when** $y(0) = 1, h = 0 \cdot 1, 0 \le x \le 0 \cdot 5$.

(Bhopal 2013; Ujjain 13; 15; Gwalior 14)

(b) $\frac{dy}{dx} = \frac{y - x}{y + x}$; $y = 1$ **for** $x = 0$, y **for** $x = 0 \cdot 1$. **(Gwalior 2012; Indore 13)**

(c) $\frac{dy}{dx} = x + y$; $y = 1$ **for** $x = 0$; **find** y **for** $x = 0 \cdot 1, 0 \cdot 2$, **check the result with exact value.**

(Indore 2012)

Use Picard's successive approximation method for solving the above questions.

Solution : (a) We have

$$\frac{dy}{dx} = f(x, y) = 1 + xy, \text{ and } x_0 = 0, y_0 = 1$$

First Approximation : $y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx = 1 + \int_0^x (1 + x) dx$

$$= 1 + x + \frac{1}{2}x^2, \qquad (\because y_0 = 1)$$

Second Approximation : $y^{(2)} = y_0 + \int_0^x f(x, y^{(1)}) dx$

$$= 1 + \int_0^x \left\{1 + x\left(1 + x + \frac{1}{2}x^2\right)\right\} dx$$

$$= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}\right)$$

Third Approximation : $y^{(3)} = 1 + \int_0^x \{1 + x(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8})\} dx$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}.$$

Tabulating, we obtain the value of y for the value of $x = 0, 0 \cdot 1, 0 \cdot 2, 0 \cdot 3, 0 \cdot 4$ and $0 \cdot 5$

x:	0	0·1	0·2	0·3	0·4	0·5
$y^{(1)}$:	1	1·105	1·22	1·345	1·48	1·625
$y^{(2)}$:	1	1·10535 = 1·105	1·22287 = 1·223	1·35501 = 1·355	1·50453 = 1·505	1·67448 = 1·675
$y^{(3)}$:	1	1·10535 = 1·105	1·22289 = 1·223	1·35519 = 1·355	1·50530 = 1·505	1·67689 = 1·677
y:	1·000	1·105	1·223	1·355	1·505	1·677

Ans.

(b) Here we have

$$\frac{dy}{dx} = \frac{y - x}{y + x} \Rightarrow f(x, y) = \frac{y - x}{y + x}, x_0 = 0, y_0 = 1$$

Now first approximation :

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx = 1 + \int_0^x \frac{1 - x}{1 + x} dx$$

$$= 1 + [2\log(1 + x) - x]_0^x$$

$$= 1 - x + 2\log(1 + x)$$

Now, $y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx = 1 + \int_0^x \frac{(1 - x + 2\log x) - x}{(1 - x + 2\log x) + x} dx$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log x}\right] dx$$

$$= 1 + x - 2\int_0^x \frac{x\, dx}{1 + 2\log(1 + x)}$$

$$= 1 + x - 2\int_0^t \frac{(e^t - 1)e^t dt}{1+2t}, \qquad [\text{where } t = \log(1+x)]$$

$$= 1 + x - 2\int_0^t \frac{e^{2t} dt}{1+2t} + 2\int_0^t \frac{e^t dt}{1+2t}$$

Which is not easily integrable.

∴ We can get the value of y, only by first approx.

Hence, at $x = 0{\cdot}1$, $y = y^{(1)} = 1 - 0{\cdot}1 + 2\log(1{\cdot}1) \Rightarrow y = 1{\cdot}0906$. **Ans.**

(c) $\frac{dy}{dx} = x + y \Rightarrow f(x,y) = x + y,\ x_0 = 0,\ y_0 = 1.$

First approximation : $y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$

$$= 1 + \int_0^x (x + y_0)dx = \int_0^x (x+1)dx$$

$$= 1 + \int_0^x (1+x)dx = 1 + x + \frac{1}{2}x^2$$

$$= 1 + 0{\cdot}1 + \frac{0{\cdot}01}{2} = 1{\cdot}105$$

Second approximation : $y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$

$$= 1 + \int_0^x \left\{ x + \left(1 + x + \frac{1}{2}x^2\right)\right\} dx$$

$$= 1 + \int_0^x \left(1 + 2x + \frac{1}{2}x^2\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

$$y^{(2)}(0{\cdot}1) = 1 + 0{\cdot}1 + 0{\cdot}01 + \frac{0{\cdot}001}{6} = 1{\cdot}1117$$

$$y^{(2)}(0{\cdot}2) = 1{\cdot}24133.$$

Third approximation : $y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})dx$

$$= 1 + \int_0^x \left[x + (1 + x + x^2 + \frac{x^3}{6}) \right] dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y^{(3)}(0{\cdot}1) = 1{\cdot}11034$$

$$y^{(3)}(0{\cdot}2) = 1{\cdot}24273$$

Fourth approximation :

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)})dx = 1 + \int_0^x [x + (1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24})]dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

$$y^{(4)}(0{\cdot}1) = 1{\cdot}11034,$$

$$y^{(4)}(0{\cdot}2) = 1{\cdot}24280$$

Exact value

Hence solution is $\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$

$$ye^{-x} = \int xe^{-x}dx + c$$

$$= -e^{-x}(1+x) + c$$

$$\Rightarrow \quad y = -x - 1 + c$$

We have $y = 0$ on $x = 0$

$$\therefore \quad 1 = -0 - 1 + c \ \Rightarrow\ c = 2$$

∴ We have the exact solution :

$$y = -x - 1 + 2e^x$$

At $x = 0{\cdot}1$, $y = -0{\cdot}1 - 1 + 2e^{0{\cdot}1} = 1{\cdot}11034$

At $x = 0{\cdot}2$, $y = -0{\cdot}2 - 1 + 2e^{0{\cdot}2} = 1{\cdot}2428$

Above results shows that the approximation obtained for $x = 0{\cdot}1$ is correct five decimal places while the approximation for $x = 0{\cdot}2$ is correct to four decimal places. **Ans.**

Q. 5. By using Picard's method. Find the value of $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$.

(Rewa 2017; Bhopal 22)

Solution : Here $x_0 = 0, y_0 = 1$ then by Picard's method,

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

$$= 1 + \int_0^x (x^2 + y^2)dx \qquad ...(1)$$

$$y^{(1)} = 1 + \int_0^x (1 + x^2)dx = 1 + x + \frac{x^3}{3} \qquad ...(2)$$

$$y^{(2)} = 1 + \int_0^x \left[x^2 + \left(1 + x + \frac{x^3}{3}\right)^2 \right] dx$$

$$= 1 + \left[x + \frac{2x^3}{3} + \frac{x^7}{63} + x^2 + \frac{2}{15}x^5 + \frac{1}{6}x^4 \right]$$

$$= 1 + x + x^2 + \frac{2}{3}x^3 + \frac{1}{6}x^4 + \frac{2}{15}x^5 + \frac{1}{63}x^7 \qquad ...(3)$$

To find the value of $y^{(3)}$ is very difficult. Hence $y^{(2)}$ is the approximate value given by eqn. (3). **Ans.**

Q. 6. Use Picard's method to approximate the value of y when $x = 0{\cdot}1$ given $\frac{dy}{dx} = 3x + y^2$, $y(0) = 1$. **(Rewa 2017)**

Solution : Let $f(x, y) = 3x + y^2$ and at $x_0 = 0$, $y_0 = 1$ is given By picard's method,

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

Hence, **first approximation :**

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx = 1 + \int_0^x (3x + y_0^2)dx$$

$$= 1 + \int_0^x (3x+1)dx = \frac{3x^2}{2} + x + 1.$$

Second approximation :

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

$$= 1 + \int_0^x [3x + \{y^{(1)}\}^2]dx$$

$$= 1 + \int_0^x \left(\frac{9}{4}x^4 + 3x^3 + 4x^2 + 5x + 1\right)dx$$

$$= \frac{9}{20}x^5 + \frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{5}{2}x^2 + x + 1.$$

Third approximation :

$$y^{(3)} = y_0 = 1 + \int_0^x [3x + \{y^{(2)}\}^2]dx$$

$$= 1 + \int_0^x \left(\frac{81}{400}x^{10} + \frac{27}{40}x^9 + \frac{141}{80}x^8 + \frac{17}{4}x^7 + \frac{1157}{180}x^6 + \frac{136}{15}x^5 + \frac{125}{12}x^4 - \frac{23}{3}x^3 + 6x^2 + 5x + 1\right)dx$$

$$= \frac{81}{4400}x^{11} + \frac{27}{400}x^{10} + \frac{47}{240}x^9 + \frac{17}{32}x^8 + \frac{1157}{1260}x^7 + \frac{62}{45}x^6 + \frac{26}{12}x^5 + \frac{23}{12}x^4 + 2x^3 + \frac{5}{2}x^2 + x + 1$$

$= 1{\cdot}12721$, at $x = 0{\cdot}1$

When $x = 0{\cdot}1$, then we get

$$y_0 = 1, y^{(1)} = 1{\cdot}115; y^{(2)} = 1{\cdot}1264; y^{(3)} = 1{\cdot}12721$$

At $x = 0{\cdot}1$, $y = 1{\cdot}127$. **Ans.**

Q. 7. Solve the following by Taylor's series method :

(a) $y' = 2xy + 1, h = 0{\cdot}2, y(0) = 0$, up to $x = 0{\cdot}4$. **(Bhopal 2013)**

(b) Solve the initial value problem by Taylor's series method for order 2 and 4.

$$y' = y - x^2 + 1, 0 \le x \le 1, y(0) = 0{\cdot}5.$$

(c) Solve $\frac{dy}{dx} = x + y$ numerically start from $y(1) = 0$ and carry to $x = 1{\cdot}2$ with $h = 0{\cdot}1$, using Taylor's series method. Compare the final result with the value of the explicit solution. **(Bhopal 2014)**

(d) $\frac{dy}{dx} = -yx^2, y(0) = 2.$ **(Ujjain 2013, Indore 13)**

Solution : (a) $x_0 = 0, y_0 = 0,$

$$y^{(i)} = 2xy + 1, y_0^{(i)} = 2 \times 0 \times 0 + 1 = 1.$$

$$y^{(ii)} = 2(xy' + y) \Rightarrow y^{(ii)} = 0$$

$$y^{(iii)} = 2(xy^{(ii)} + 2y^{(i)}) \Rightarrow y_0^{(iii)} = 4$$

$$y^{(iv)} = 2(xy^{(iii)} + 3y^{(ii)}) \Rightarrow y_0^{(iv)} = 0$$

$$y^{(v)} = 2(xy^{(iv)} + 4y^{(iii)}) \Rightarrow y_0^{(v)} = 32$$

By Taylor's series,

$$y_x = y_0 + (x - x_0)y_0^{(i)} + \frac{(x-x_0)^2}{2!}y_0^{(ii)} + \frac{(x-x_0)^3}{3!}y_0^{(iii)} + \frac{(x-x_0)^4}{4!}y_0^{(iv)} + \frac{(x-x_0)^5}{5!}y_0^{(v)} + \ldots$$

Taking $x = 0{\cdot}2$,

$$y(0{\cdot}2) = 0 + (0{\cdot}2)1 + 0 + \frac{(0{\cdot}2)^3}{3!} \times 4 + \frac{(0{\cdot}2)^5}{5!} \times 32$$

$$= 0{\cdot}20535466 \approx 0{\cdot}21$$

Now, $x_1 = 0{\cdot}2$ and $y_1 = 0{\cdot}21$ evaluate $y(0{\cdot}4)$, we get :

$$y_1^{(i)} = 2(0{\cdot}2)(0{\cdot}21) + 1 = 1{\cdot}084$$

$$y_1^{(ii)} = 2[0{\cdot}2 \times 1{\cdot}084 + 0{\cdot}21] = 0{\cdot}8536$$

$$y_1^{(iii)} = 2[0{\cdot}2 \times (0{\cdot}8536) + 2 \times 1{\cdot}084] = 4{\cdot}67744$$

$$y_1^{(iv)} = 2[0{\cdot}2 \times (4{\cdot}67744) + (3 \times 0{\cdot}8536)] = 6{\cdot}992576$$

$$y_1^{(v)} = 2[0{\cdot}2 \times (6{\cdot}992576) + 4 \times 4{\cdot}67744] = 40{\cdot}21655$$

Put the value of y_1 and its derivatives in Taylor series expansion we get,

$$y(x_2) = y(0{\cdot}4) = y_1 + (x_2 - x_1)y_1^{(i)} + \frac{(x_2-x_1)^2}{2!}y_1^{(ii)} + \frac{(x_2-x_1)^3}{3!}y_1^{(iii)} + \frac{(x_2-x_1)^4}{4!}y_1^{(iv)} + \frac{(x_2-x_1)^5}{5!}y_1^{(v)}$$

$$= 0\cdot21 + 0\cdot2\times(1\cdot084) + \frac{(0\cdot2)^2}{2!}(0\cdot8536) + \frac{(0\cdot2)^3}{3!}(4\cdot67744) + \frac{(0\cdot2)^4}{4!}(6\cdot992576) + \frac{(0\cdot2)^5}{5!}(40\cdot21655)$$

$$= 0\cdot45068 \approx 0\cdot451.$$ **Ans.**

(b) $y' = y - x^2 + 1,\ 0 \le x \le 1,\ y(0) = 0\cdot5$

Let y is a function of x *i.e.*, $y(x)$ and differentiate successive w.r.t. 'x',

$$f(x, y(x)) = y(x) - x^2 + 1$$

$$f'(x, y(x)) = y' - 2x = (y - x^2 + 1) - 2x$$

$$f''(x, y(x)) = y' - 2x - 2 = (y - x^2 + 1) - 2x - 2$$

$$= y - x^2 - 2x - 1$$

$$f'''(x, y(x)) = y' - 2x - 2 = y - x^2 + 1 - 2x - 2$$

$$= y - x^2 - 2x - 1$$

$\therefore$ Taylor expansion of order 2 is

$$T^{(2)}(x_i, y_i) = f(x, y) + \frac{h}{2}f'(x_i, y_i)$$

$$= y_i - x_i^2 + 1 + \frac{h}{2}(y_i - x_i^2 - 2x_i + 1)$$

$$= (1 + \frac{h}{2})(y_i - x_i^2 + 1) - hx_i$$

and Taylor's expansion of order 4 is

$$T^{(4)}(x_i, y_i) = f(x_i, y_i) + \frac{h}{2}f'(x_i, y_i) + \frac{h^2}{3!}f''(x_i, y_i) + \frac{h^3}{4!}f'''(x_i, y_i)$$

$$= y_i - x_i^2 + 1 - \frac{h}{2}(y_i - x_i^2 - 2x_i + 1) + \frac{h^2}{6}(y_i - x_i^2 - 2x_i - 1) + \frac{h^3}{24}(y_i - x_i^2 - 2x_i - 1)$$

$$= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(y_i - x_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(hx_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}$$

By Taylor's expansion of order 2,

$$y_0 = 0\cdot5$$

$$y_{i+1} = y_i + h\left[\left(1 + \frac{h}{2}\right)(y_i - x_i^2 + 1) - hx_i\right], \qquad [\because\ y_{n+1} = y_n + hT^{(k)}(x, y)]$$

Taking $n = 0\cdot2$, $h = \frac{b-a}{N} \Rightarrow 0\cdot2 = \frac{1-0}{N} \Rightarrow N = 5$

$$x_i = a + ih = 0 + ih.$$

$$\therefore \quad y_{i+1} = y_i + (0\cdot2)\left[\left(1 + \frac{0\cdot2}{2}\right)(y_i - 0\cdot04i^2 + 1) - 0\cdot04i\right]$$

$$= 1\cdot22y_i - 0\cdot0088i^2 - 0\cdot008i + 0\cdot22$$

By Taylor's expansion of order 4,

$$y_{i+1} = y_i + 0\cdot2\left[1 + \frac{0\cdot2}{2} + \frac{0\cdot04}{6} + \frac{0\cdot008}{24}(y_i - 0\cdot04i^2) - \left(1 + \frac{0\cdot2}{3} + \frac{0\cdot04}{3}\right)(0\cdot04i) + 1 + \frac{0\cdot2}{2} + \frac{0\cdot04}{6} - \frac{0\cdot008}{24}\right]$$

$$= 1\cdot221y_i - 0\cdot008856i + 0\cdot2186$$

$\forall\ i = 0, 1, .. 4$

Exact solution of equation $y(x) = (x+1)^2 - 0\cdot5e^x$.

$\therefore$ Following table shows approximation with errors,

x_i	Exact $y(x_i)$	By Taylor exp. of order 2	Error $\lvert y(x_i) - y_i \rvert$ from y_i	By Taylor exp. of order 4	Error $\lvert y(x_i) - y_i \rvert$ from y_i
0·0	0·5	0·5	0	0·5	0
0·2	0·8292986	0·83	0·0007014	0·8293	$1\cdot4\times10^{-6}$
0·4	1·2140817	1·2158	0·0017123	1·214091	$3\cdot4\times10^{-6}$
0·6	1·6489406	1·652076	0·0031354	1·6489468	$6\cdot2\times10^{-6}$
0·8	2·1272295	2·1323327	0·0051032	2·1272396	$1\cdot01\times10^{-5}$
1·0	2·6408591	2·6486459	0·0077868	2·6408744	$1\cdot53\times10^{-5}$

Ans.

(c) Given, $\frac{dy}{dx} = f(x, y) = x + y, x_0 = 1, y_0 = 0$

$$\therefore \quad y^{(i)} = x + y \qquad \ldots(1)$$

Successive differentiation of eqn. (1),

$y^{(ii)} = 1 + y^{(i)}$, Similarly $y^{(iii)} = y^{(ii)}, y^{(iv)} = y^{(iii)}$....

Put $x = 1$,

$y_0^{(i)} = 1, y_0^{(ii)} = 2 = y_0^{(iii)} = y_0^{(iv)} = y_0^{(v)}$,.....

Let y is a function of x then by Taylor's expansion,

$$y(x) = y_0 + (x - x_0)y_0^{(i)} + \frac{(x - x_0)^2}{2!}y_0^{(ii)} + \ldots\ldots, \text{ where, } x - x_0 = h$$

$$\therefore \quad y_1 = y_0 + hy_0^{(i)} + \frac{h^2}{2!}y_0^{(ii)} + \frac{h^3}{3!}y_0^{(iii)} + \ldots..$$

Where $h = 0\cdot1, x = x_0 + h = 1 + 0\cdot1$

$$\Rightarrow y_1 = y(1\cdot1) = 0 + (0\cdot1)\times1 + \frac{(0\cdot1)^2}{2!}\times2 + \frac{(0\cdot1)^3}{3!}\times2 + \frac{(0\cdot1)^4}{4!}\times2 + \frac{(0\cdot1)^5}{5!}\times2$$

$$= 0\cdot1103$$

Again, $y_2 = y(1\cdot2) = y_1 + hy_1^{(i)} + \frac{h^2}{2!}y_1^{(ii)} + \frac{h^3}{3!}y_1^{(iii)} + \ldots\ldots$

In $y_1 = 0\cdot1103,$

$y_1^{(i)} = x_1 + y_1 = 1\cdot1 + 0\cdot1103 = 1\cdot2103$

$y_1^{(ii)} = 1 + y_1^{(i)} = 2\cdot2103$

$y_1^{(iii)} = 2\cdot2103 = y_1^{(iv)} = y_1^{(v)} = \ldots\ldots\ldots\ldots,$

$$y_2 = y(1\cdot2) = 0\cdot1103 + (0\cdot1)(1\cdot2103) + \frac{(0\cdot1)^2}{2!}(2\cdot2103) + \frac{(0\cdot1)^3}{3!}(2\cdot2103) + \frac{(0\cdot1)^4}{4!}(2\cdot2103) + \frac{(0\cdot1)^5}{5!}(2\cdot2103)$$

$= 0\cdot242$ (approx.) **Ans.**

Exact solution is : $y = -x - 1 + 2e^{x-1}$

Here put $x = 1\cdot2,$

$y = -1\cdot2 - 1 + 2e^{1\cdot2-1}$

$= -2\cdot2 + 2(1\cdot221) = 0\cdot242.$

(d) Given, $y^{(i)} = -yx^2,$

$x_0 = 0, y_0 = 2$

Successive differentiation of eqn. (1),

$y^{(ii)} = -y^{(i)}x^2 - 2xy,$

$y^{(iii)} = -y^{(ii)}x^2 - 4xy^{(i)} - 2y,$

$y^{(iv)} = -y^{(iii)}x^2 - 6xy^{(ii)} - 6y^{(i)}$

$y^{(v)} = -y^{(iv)}x^2 - 8xy^{(iii)} - 12y^{(ii)}$

$y^{(vi)} = -y^{(v)}x^2 - 10xy^{(iv)} - 20y^{(iii)}$

Put $x = 0, y = 2,$

$y_0^{(i)} = 0, y_0^{(ii)} = 0, y_0^{(iii)} = -4, y_0^{(iv)} = 0, y_0^{(v)} = 0, y_0^{(vi)} = 80$

Now $x = x_0 = 0$ (by Taylor's expansion)

$$y(x) = y_0 + (x-0)y_0^{(i)} + \frac{(x-0)^2}{2!}y_0^{(ii)} + \ldots.$$

$$= y_0 + xy_0^{(i)} + \frac{x^2}{2!}y_0^{(ii)} + \frac{x^3}{3!}y_0^{(iii)} + \frac{x^4}{4!}y_0^{(iv)} + \ldots. \quad \ldots(2)$$

$= 2 - \frac{4}{3!}x^3 + \frac{80}{6!}x^6 + \ldots\ldots$(put value $y_0, y_0^{(i)}, y_0^{(ii)}$ ....in eqn. (2))

Ans.

Q. 8. Find the solution of ODE (ordinary differential equation) :

(Gwalior 2014; Jabalpur 15)

(a) $\frac{dy}{dx} = xy,\ y(0) = 1, h = 0\cdot1$ **to** $y(0\cdot4)$. **(Jabalpur 2011, 12, 14; Indore 13, 14, 16; Chhindwara 22; Ujjain 22)**

(b) $\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1,$ **for** $x = 0\cdot1$ **find the value of** y **by Euler's method up to five approx.**

(c) $\frac{dy}{dx} = x - y^2,\ y(0) = 1,$ **for** $y(0\cdot1)$ **up to four approx.**

(d) Apply Euler's method solve differential equation.

(Gwalior 2011; Ujjain 13; Jabalpur 16)

$\frac{dy}{dx} = -\frac{y^2}{1+x}, y(0) = 1$ **at** $h = 0\cdot05$ **find** $y(0\cdot2)$.

Solution : (a) Given : $\frac{dy}{dx} = xy,$

According to question $f(x,y) = xy,\ x_0 = 0, y(0) = 1, h = 0\cdot1,$ and $m = 0\cdot4$

$\therefore$ By Euler's approx. $y_m = y_{m-1} + hf(x_{m-1} - x_{m-1})$

Put the value of $m = 0, 1, 2, 3.$

$y_1 = y_0 + hf(x_0, y_0) = 1 + 0\cdot1[0\times1] = 1.$

$y_2 = y_1 + hf(x_1, y_1) = 1 + 0\cdot1(0\cdot1\times1) = 1\cdot01,$ $(\because x_1 = x_0 + h)$

$y_3 = y_2 + hf(x_2, y_2) = 1\cdot01 + 0\cdot1(0\cdot2\times1\cdot01) = 1\cdot0302$

$y_4 = y_3 + hf(x_3, y_3) = 1\cdot0302 + 0\cdot1(0\cdot3\times1\cdot0302) = 1\cdot061106.$ **Ans.**

(b) Given :

$\frac{dy}{dx} = \frac{y-x}{y+x} \Rightarrow f(x,y) = \frac{y-x}{y+x}$

$x_0 = 0, y_0 = 1, m = 5, h = \frac{x_m - x_0}{m} = \frac{0\cdot1-0}{5} = 0\cdot02.$

At first approximation $x = 0\cdot02,$

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0\cdot02\times\frac{1-0}{1+0} = 1\cdot02$$

$$y_2 = y_1 + hf(x_1, y_1) = 1\cdot02 + 0\cdot02\times\frac{1\cdot02-0\cdot02}{1\cdot02+0\cdot02}$$

$$= 1\cdot02 + 0\cdot02\times\frac{1}{1\cdot04} = 1\cdot0392$$

$$y_3 = y_2 + hf(x_2, y_2) = 1\cdot0392 + 0\cdot02\times\frac{1\cdot0392-0\cdot04}{1\cdot0392+0\cdot04}$$

$$= 1\cdot039 + 0\cdot02\times\frac{0\cdot9992}{1\cdot0792} = 1\cdot0577$$

$$y_4 = y_3 + hf(x_3, y_3) = 1\cdot0577 + 0\cdot02\times\frac{1\cdot0577-0\cdot06}{1\cdot0577+0\cdot06}$$

$$= 1\cdot0577 + 0\cdot02\times(0\cdot893) = 1\cdot0756.$$

and at $x_5 = 1$ fifth approximation is given by

$$y_5 = y_4 + hf(x_4, y_4) = 1{\cdot}0756 + 0{\cdot}02 \times \frac{1{\cdot}0756 - 0{\cdot}08}{1{\cdot}0756 + 0{\cdot}08}$$

$$= 1{\cdot}0756 + 0{\cdot}02 \times \frac{0{\cdot}9956}{1{\cdot}1556}$$

$$= 1{\cdot}0756 + 0{\cdot}172 = 1{\cdot}0928.$$

Ans.

(c) Here, $\frac{dy}{dx} = x - y^2 = f(x, y)$.

$x_0 = 0, y_0 = 1, h = \frac{0{\cdot}1 - 0}{4} = 0{\cdot}025$

By Euler method $y_{m+1} = y_m + hf(x_m, y_m)$

$y_1 = y_0 + h(x_0 - y_0^2) = 1 + 0{\cdot}025(0 - 1^2) = 0{\cdot}9750$

$y_2 = y_1 + h(x_1 - y_1^2) = 0{\cdot}9750 + 0{\cdot}025\,[0{\cdot}025 - (0{\cdot}9750)^2] = 0{\cdot}9519$

$y_3 = y_2 + h(x_2 - y_2^2) = 0{\cdot}9519 + 0{\cdot}025[0{\cdot}050 - (0{\cdot}9519)^2] = 0{\cdot}9305$

$y_4 = y_3 + h(x_3 - y_3^2) = 0{\cdot}9305 + 0{\cdot}025[0{\cdot}075 - (0{\cdot}9305)^2] = 0{\cdot}9107$

The above result may be tabulated as follows :

m	x_m	y_m	y_m^2	$f(x_m, y_m)$	$hf(x_n, y_n)$	$y_{m+1} = y_m + hf(x_m, y_m)$
0	0·000	1·0	1·0000	–1·0000	–0·025	$y_1 = 0{\cdot}9750$
1	0·025	$y_1 = 0{\cdot}9750$	0·9506	–0·9256	–0·0231	$y_2 = 0{\cdot}9519$
2	0·050	$y_2 = 0{\cdot}9519$	0·9061	–0·8561	–0·0214	$y_3 = 0{\cdot}9305$
3	0·075	$y_3 = 0{\cdot}9305$	0·8658	–0·7908	–0·0198	$y_4 = 0{\cdot}9107$
4	0·100	$y_4 = 0{\cdot}9107$				

Ans.

(d) $\frac{dy}{dx} = f(x, y) = -\frac{y^2}{1+x}$

$y_{m+1} = y_m + hf(x_m, y_m)$

$x_0 = 0, y_0 = 1, h = 0{\cdot}05$ and $x_m = 0{\cdot}2 = x_0 + 4h$

$m = 0, 1, 2, 3$ now approx. are

$$y_1 = y_0 + hf(x_0, y_0) = 1 - 0{\cdot}05 \times \frac{1^2}{1+0} = 0{\cdot}95$$

$$y_2 = y_1 + hf(x_1, y_1) = 0{\cdot}95 - 0{\cdot}05 \times \frac{(0{\cdot}95)^2}{1+0{\cdot}05} = 0{\cdot}90702$$

$$y_3 = y_2 + hf(x_2, y_2) = 0{\cdot}90702 - 0{\cdot}05 \times \frac{(0{\cdot}90702)^2}{1+0{\cdot}1}$$

$$= 0{\cdot}86963$$

$$y_4 = y_3 + hf(x_3, y_3) = 0{\cdot}86963 - 0{\cdot}05 \times \frac{(0{\cdot}86963)^2}{1+0{\cdot}15}$$

$$= 0{\cdot}83675$$

Now we know exact solution is

$$\frac{dy}{dx} = -\frac{y^2}{1+x} \Rightarrow -\int \frac{1}{y^2}dy = \int \frac{1}{1+x}dx$$

$$\Rightarrow \quad y^{-1} = \log(1+x) + A \Rightarrow y = \frac{1}{A + \log(1+x)}$$

$$y_0 = \frac{1}{A + \log(1+x_0)} \Rightarrow 1 = \frac{1}{A} \Rightarrow A = 1.$$

$$\therefore \quad y = \frac{1}{1 + \log(1+x)} \Rightarrow y(0{\cdot}2) = 0{\cdot}84579$$

$\therefore$ Error at $y\,(0{\cdot}2)$ is

$$= 0{\cdot}84579 - 0{\cdot}83675$$

$$= 0{\cdot}00904.$$

Ans.

Q. 9. Using Euler's method calculate the solution of $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$. For y at $x = 1$ upto five. **(Gwalior 2016; Jabalpur 22)**

Solution : $f(x, y) = x^2 + y^2, x_0 = 0, y_0 = 1$ and $n = 5, h = \frac{x - x_0}{n} \Rightarrow h = \frac{1-0}{5} = 0{\cdot}2$

$$x_1 = x_0 + h = 0{\cdot}2, x_2 = 0{\cdot}4, x_3 = 0{\cdot}6, x_4 = 0{\cdot}8, x_5 = 1{\cdot}0$$

Now, by Euler's method,

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \text{ where } n = 1, 2, 3, 4, 5$$

$\therefore$ $y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0^2 + y_0^2)$

$$= 1 + (0{\cdot}2)(0+1)^2 = 1{\cdot}02$$

$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(x_1^2 + y_1^2)$

$$= 1{\cdot}02 + (0{\cdot}2)[(0{\cdot}2)^2 + (1{\cdot}02)^2] = 1{\cdot}23608$$

$y_3 = y_2 + hf(x_2, y_2) = y_2 + h(x_2^2 + y_2^2)$

$$= 1{\cdot}23608 + (0{\cdot}2)[(0{\cdot}4)^2 + (1{\cdot}23608)^2]$$

$$= 1{\cdot}57366$$

$y_4 = y_3 + hf(x_3, y_3) = y_3 + h(x_3^2 + y_3^2)$

$$= 1{\cdot}57366 + (0{\cdot}2)[(0{\cdot}6)^2 + (1{\cdot}57366)^2]$$

$$= 2{\cdot}14094$$

$y_5 = y_4 + hf(x_4, y_4) = y_4 + h(x_4^2 + y_4^2)$

$$= 2{\cdot}14094 + (0{\cdot}2)[(0{\cdot}8)^2 + (2{\cdot}14094)^2]$$

$$= 3{\cdot}18566$$

Hence, $y_5 = y(x_5) = y(1) = 3{\cdot}18566.$

Ans.

Q. 10. Solve the differential equation in six steps using Euler's method :

$$\frac{dy}{dx} = x + y, y(0) = 0, h = 0{\cdot}2.$$ **(Jabalpur 2022; Bhopal 22)**

Solution : Here, $f(x, y) = x + y, x_0 = 0, y_0 = 0, h = 0{\cdot}2$ and $n = 6$.

Now, by Euler's method, we have

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \text{ where } n = 1,2,3,4,5,6$$

Hence, $x_1 = x_0 + h = 0{\cdot}2, x_2 = 0{\cdot}4, x_3 = 0{\cdot}6, x_4 = 0{\cdot}8, x_5 = 1, x_6 = 1{\cdot}2$

$\therefore$ $y_1 = y_0 + hf(x_0, y_0) = 0 + (0{\cdot}2)(0 + 0) = 0$

$y_2 = y_1 + hf(x_1, y_1) = 0 + (0{\cdot}2)(0{\cdot}2 + 0) = 0{\cdot}04$

$y_3 = y_2 + hf(x_2, y_2) = 0{\cdot}04 + (0{\cdot}2)(0{\cdot}4 + 0{\cdot}04) = 0{\cdot}128$

$y_4 = y_3 + hf(x_3, y_3) = 0{\cdot}128 + (0{\cdot}2)(0{\cdot}6 + 0{\cdot}128) = 0{\cdot}2736$

$y_5 = y_4 + hf(x_4, y_4) = 0{\cdot}2736 + (0{\cdot}2)(0{\cdot}8 + 0{\cdot}2736) = 0{\cdot}48832$

$y_6 = y_5 + hf(x_5, y_5) = 0{\cdot}48832 + (0{\cdot}2)(1{\cdot}0 + 0{\cdot}48832) = 0{\cdot}785984$

Hence $y_6 = y(1{\cdot}2) = 0{\cdot}785984$. **Ans.**

Q. 11. Given $\frac{dy}{dx} = x + y$ determine the value of y at $x = 0{\cdot}05$ and $x = 0{\cdot}1$ with initial condition $y(0) = 1$. (Upto four decimal places) **(Gwalior 2022)**

Solution : The given equation is

$$\frac{dy}{dx} = x + y = f(x, y) \text{ (let)}$$

and $x_0 = 0, y_0 = 1$

Hence, by Euler's method, at $x = 0{\cdot}05$

$$y_1^{(1)} = y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0) = 1 + (0{\cdot}05)(0 + 1) = 1{\cdot}05$$

Hence, by modified Euler's method,

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(1)})]$$

$$= 1 + \frac{0{\cdot}05}{2}[(0 + 1) + (0{\cdot}05) + (1{\cdot}05)] = 1{\cdot}0525$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(2)})]$$

$$= 1 + \frac{0{\cdot}05}{2}[(0 + 1) + (0{\cdot}05) + 1{\cdot}05256] = 1{\cdot}05256$$

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})] = y_0 + \frac{h}{2}[(x_0 + y_0) + (x_1 + y_1^{(3)})]$$

$$= 1 + \frac{0{\cdot}05}{2}[(0 + 1) + (0{\cdot}05 + 1{\cdot}0256)] = 1{\cdot}05256$$

$\therefore$ $y_1^{(3)} = y_1^{(4)}$

Hence $x_1 = 0{\cdot}05$, $y_1 = 1{\cdot}0526$

Now, $x = 0{\cdot}1$,

$$y_2^{(1)} = y_1 + hf(x_1, y_1) = y_1 + h[x_1 + y_1]$$

$$= 1{\cdot}0526 + (0{\cdot}05)[0{\cdot}05 + 1{\cdot}0526] = 1{\cdot}1077$$

Now, by Euler's modified method,

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = y_1 + \frac{h}{2}[(x_1 + y_1) + (x_2 + y_2^{(1)})]$$

$$= 1{\cdot}0526 + \frac{(0{\cdot}05)}{2}[(0{\cdot}05 + 1{\cdot}0526) + (0{\cdot}1 + 1{\cdot}1077)] = 1{\cdot}1104$$

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1{\cdot}0526 + (0{\cdot}025)[1{\cdot}1026 + (0{\cdot}1 + 1{\cdot}1104)] = 1{\cdot}1104$$

$\therefore$ $y_2^{(2)} = y_2^{(3)} = 1{\cdot}1104$

Hence, at $x = 0{\cdot}1$ value of y is $1{\cdot}1104$. **Ans.**

Q. 12. Using Euler's method, solve the equation at $x = 0{\cdot}5$ in the range $0 \le x \le 0{\cdot}5$ by taking $h = 0{\cdot}1$

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0.$$ **(Bhopal 2016; Gwalior 17)**

Solution : $\because$ The range of interval is $x = 0$ to $x = 0{\cdot}5$ divided into 5 steps.

Hence, we will calculate y at $x_1 = 0{\cdot}1, x_2 = 0{\cdot}2, x_3 = 0{\cdot}3, x_4 = 0{\cdot}4, x_5 = 0{\cdot}5$

Hence, by Euler's method, we have

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

at $x = 0{\cdot}1$ the first approximation :

$$y_1 = y_0 + hf(x_0, y_0)$$

where $x_0 = 0, y_0 = 0$, $f(x_0, y_0) = x_0^2 + y_0^2 = 0$

$$y_1 = 0 + 0{\cdot}1 \times 0 = 0$$

Second approximation of y at $x_2 = 0{\cdot}2$

$$y_2 = y_1 + hf(x_1, y_1), \text{ where } x_1 = 0{\cdot}1, y_1 = 0 \text{ and}$$

$f(x_1, y_1) = x_1^2 + y_1^2$

$$y_2 = 0 + 0{\cdot}1[(0{\cdot}1)^2 + 0^2] = 0{\cdot}001$$

Third approximation of y at $x_3 = 0{\cdot}3$

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + h[x_2^2 + y_2^2]$$

$$y_3 = 0{\cdot}001 + 0{\cdot}1[(0{\cdot}2)^2 + (0{\cdot}001)^2] = 0{\cdot}005$$

Fourth approximation of y at $x_4 = 0{\cdot}4$

$$y_4 = y_3 + hf(x_3, y_3) = y_3 + h[x_3^2 + y_3^2]$$

$$y_4 = 0{\cdot}005 + 0{\cdot}1[(0{\cdot}3)^2 + (0{\cdot}005)^2] = 0{\cdot}014$$

Fifth approximation of y at $x_5 = 0{\cdot}5$

$$y_5 = y_4 + hf(x_4 + y_4) = y_4 + h[x_4^2 + y_4^2]$$

$$= 0{\cdot}014 + 0{\cdot}1[(0{\cdot}4)^2 + (0{\cdot}014)^2]$$

$$= 0{\cdot}014 + 0{\cdot}016 = 0{\cdot}030.$$ Ans.

Q. 13. Using modified Euler's method, find solution of equation

$\frac{dy}{dx} = 1 - y = f(x, y)$ **with $y(0) = 0$ in the range $0 \le x \le 0{\cdot}3$ by taking $h = 0{\cdot}1$.**

(Indore 2017, Chhindwara 22)

Solution : Here, $\frac{dy}{dx} = 1 - y$

$\Rightarrow$ $f(x, y) = 1 - y$ with $y(0) = 0$

i.e., $x_0 = 0$, $y_0 = 0$, and $h = 0{\cdot}1$

By Euler's method,

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h(1 - y_0) = 0 + (0{\cdot}1)(1 - 0) = 0{\cdot}1$$

$\Rightarrow$ $y(0{\cdot}1) = 0{\cdot}1$

On $x_2 = 0{\cdot}2$,

$$y_2 = y_1 + hf(x_1, y_1) = y_1 + h(1 - y_1)$$

$$= 0{\cdot}1 + (0{\cdot}1)(1 - 0{\cdot}1) = 0{\cdot}19 \Rightarrow y(0{\cdot}2) = 0{\cdot}19$$

On $x_3 = 0{\cdot}3$,

$$y_3 = y_2 + hf(x_2, y_2) = y_2 + h(1 - y_2)$$

$$= 0{\cdot}19 + (0{\cdot}1)(1 - 0{\cdot}19) = 0{\cdot}271$$

$\Rightarrow$ $y(0{\cdot}3) = 0{\cdot}271$. Ans.

Q. 14. Ue Runge-Kutta method to find y when $x = 1{\cdot}1$ in stpes of $0{\cdot}1$, given that $\frac{dy}{dx} = x^2 + y^2, y(1) = y(1{\cdot}5)$. **(Indore 2017; Jabalpur 22)**

Solution : Here $f(x, y) = x^2 + y^2$, and $x_0 = 1$, $y_0 = 1{\cdot}5$, $h = 0{\cdot}1$.

For the first interval, we have

$$K_1 = hf(x_0, y_0) = 0{\cdot}1f(1, 1{\cdot}5) = (0{\cdot}1)[1^2 + (1{\cdot}5)^2]$$

$$= 0{\cdot}325$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = (0{\cdot}1)f(1{\cdot}05, 1{\cdot}6625)$$

$$= (0{\cdot}1)[(1{\cdot}05)^2 + (1{\cdot}6625)^2] = 0{\cdot}38664$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = (0{\cdot}1)f(1{\cdot}05, 1{\cdot}69332)$$

$$= (0{\cdot}1)[(1{\cdot}05)^2 + (1{\cdot}69332)^2] = 0{\cdot}39698$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = (0{\cdot}1)f(1{\cdot}1, 1{\cdot}89698)$$

$$= (0{\cdot}1)[(1{\cdot}1)^2 + (1{\cdot}89698)^2] = 0{\cdot}48085.$$

Here $x_1 = x_0 + h = 1 + 0{\cdot}1 = 1{\cdot}1$, so the value y_1 of y at $x_1 = 1{\cdot}1$ given by

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1{\cdot}5 + \frac{1}{6}(0{\cdot}325 + 0{\cdot}77328 + 0{\cdot}79396 + 0{\cdot}48085)$$

$$= 1{\cdot}895515.$$

Now we shall find y for $x_2 = x_0 + 2h = 1{\cdot}2$.

In this case, $x_1 = 1{\cdot}1, y_1 = 1{\cdot}895515, h = 0{\cdot}1$

$$K_1 = hf(x_1, y_1) = (0{\cdot}1)[(1{\cdot}1)^2 + (1{\cdot}895515)^2] = 0{\cdot}4802977$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = (0{\cdot}1)f(1{\cdot}15, 2{\cdot}13566)$$

$$= (0{\cdot}1)[(1{\cdot}15)^2 + (2{\cdot}13566)^2] = 0{\cdot}58835$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = (0{\cdot}1)f(1{\cdot}15, 2{\cdot}18969)$$

$$= (0{\cdot}1)[(1{\cdot}15)^2 + (2{\cdot}18969)^2] = 0{\cdot}61172$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = (0{\cdot}1) f(1{\cdot}2, 2{\cdot}50723)$$

$$= (0{\cdot}1)[(1{\cdot}2)^2 + (2{\cdot}50723)^2] = 0{\cdot}77262.$$

So the value y_2 of y at $x_2 = 1{\cdot}2$ is given by

$$y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1{\cdot}895515 + \frac{1}{6}(0{\cdot}4802977 + 1{\cdot}1767000 + 1{\cdot}2234400 + 0{\cdot}7726200)$$

$$= 2{\cdot}5043.$$ **Ans.**

Q. 15. By Runge-kutta's method of second and fourth order, find the value of $y(0{\cdot}1)$ and $y(0{\cdot}2)$ if $\frac{dy}{dx} = y - x$, with $y(0) = 2$.

(Bhopal 2011, 14; Gwalior 11, 13; Rewa 16)

Solution : Here given that $x_0 = 0, y_0 = 2, h = 0{\cdot}1$

(i) Second Order : Taking $h = 0{\cdot}1$

$$k_1 = hf(x_0, y_0) = (0{\cdot}1)(y_0 - x_0) = 0{\cdot}1(2 - 0) = 0{\cdot}2$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = (0{\cdot}1)[y_0 + k_1 - (x_0 + h)]$$

$$= 0{\cdot}1[2 + 0{\cdot}2 - (0 + 0{\cdot}1)]$$

$$= 0{\cdot}21$$

$$y_1 = y_0 + \left(\frac{k_1 + k_2}{2}\right) = 2 + \left(\frac{0{\cdot}2 + 0{\cdot}21}{2}\right) = 2{\cdot}205$$

For finding $y_2 = y(0{\cdot}2)$, we have $x_1 = 0{\cdot}1, y_1 = 2{\cdot}205$

Hence $k_1 = hf(x_1, y_1) = (0{\cdot}1)[y_1 - x_1] = (0{\cdot}1)[2{\cdot}205 - 0{\cdot}1] = 0{\cdot}2105$

$$k_2 = hf(x_1 + h_1, y_1 + k_1) = 0{\cdot}22155$$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 2{\cdot}205 + \frac{1}{2}(2{\cdot}2105 + 0{\cdot}22155)$$

$$= 2{\cdot}4210$$

Similarly, $y(0{\cdot}1) = 2{\cdot}205$ and $y(0{\cdot}2) = 2{\cdot}4210$. **Ans.**

(ii) Fourth Order : $k_1 = hf(x_0, y_0) = 0{\cdot}1(y_0 - x_0) = 0{\cdot}1(2 - 0) = 0{\cdot}2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[y_0 + \frac{k_1}{2} - \left(x_0 + \frac{h}{2}\right)\right]$$

$$= 0{\cdot}1\left[2 + 0{\cdot}1 - \left(0 + \frac{0{\cdot}1}{2}\right)\right]$$

$$= 0{\cdot}205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0{\cdot}1\left[2 + \frac{0{\cdot}205}{2} - \left(0 + \frac{0{\cdot}1}{2}\right)\right]$$

$$= 0{\cdot}20525$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0{\cdot}1[2 + 0{\cdot}02525 - (0 + 0{\cdot}1)]$$

$$= 0{\cdot}210525$$

Hence $$y(0{\cdot}1) = 2 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 2 + \frac{1}{6}[0{\cdot}2 + 0{\cdot}41 + 0{\cdot}82100 + 0{\cdot}210525]$$

$$= 2{\cdot}2025$$

On $x = 0{\cdot}2$ given $x_1 = 0{\cdot}1, y_1 = 2{\cdot}2052$

$$k_1 = hf(x_1, y_1) = 0{\cdot}1[y_1 - x_1] = 0{\cdot}1[2{\cdot}2052 - 0{\cdot}1] = 0{\cdot}21052$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0{\cdot}1\left[2{\cdot}2050 + \frac{0{\cdot}21052}{2} - \left(0{\cdot}1 + \frac{0{\cdot}1}{2}\right)\right]$$

$$= 0{\cdot}216046$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0{\cdot}2163223$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0{\cdot}1[(y_1 + k_3) - (x_1 + h)]$$

$$= 0{\cdot}1[(2{\cdot}2052 + 0{\cdot}2163223) - (0{\cdot}1 + 0{\cdot}1)]$$

$$= 0{\cdot}22215223$$

$$y(0{\cdot}2) = 2{\cdot}2052 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2{\cdot}2052 + \frac{1}{6}[0{\cdot}21052 + 2(0{\cdot}216046) + 2(0{\cdot}2163223) + 0{\cdot}22215223]$$

$$= 2{\cdot}4214.$$ **Ans.**

Q. 16. By using Runge-Kutta method calculate the solution of equation $\frac{dy}{dx} = \frac{1}{x + y}$, (for $x = 0{\cdot}5$). When $x_0 = 0, y_0 = 1$ by taking $h = 0{\cdot}5$.

(Rewa 2014; Ujjain 16; Jabalpur 16, 17; Chhatarpur 22)

Solution : Here $x_0 = 0, y_0 = 1, h = 0{\cdot}5$ and $f(x, y) = \frac{1}{x + y}$

$$k_1 = hf(x_0, y_0) = 0{\cdot}5\left(\frac{1}{1 + 0}\right) = 0{\cdot}5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0{\cdot}5f(0{\cdot}25, 1{\cdot}25)$$

$$= (0{\cdot}5)\left(\frac{1}{0{\cdot}25 + 1{\cdot}25}\right) = 0{\cdot}333333$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0{\cdot}5f\left(0{\cdot}25, \frac{7}{6}\right)$$

$$= 0{\cdot}5 \times \frac{1}{0{\cdot}25 + \frac{7}{6}} = 0{\cdot}35294$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0{\cdot}5f(0{\cdot}5, 1{\cdot}35294)$$

$$= 0{\cdot}5 \times \frac{1}{0{\cdot}5 + 1{\cdot}35294} = 0{\cdot}26984$$

Here $x_1 = 0 + 0{\cdot}5 = 0{\cdot}5$

Hence $x_1 = 0{\cdot}5$ then value of y

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0{\cdot}5 + 0{\cdot}6667 + 0{\cdot}7058 + 0{\cdot}2698)$$

$$= 1{\cdot}35705.$$ **Ans.**

Q. 17. Solve the following differential equation by Runge-Kutta method given with their initial condition : **(Indore 2014, Rewa 15)**

(a) $\frac{dy}{dx} = x + y, y(0) = 1, h = 0{\cdot}1,$ **with from** $x = 0$ **to** $x = 0{\cdot}4$

(Rewa 2011, 17; Jabalpur 12; Indore 13; Bhopal 16)

(b) $\frac{dy}{dx} = 1 + y^2, y(0) = 0, x = 0{\cdot}2, 0{\cdot}4, 0{\cdot}6$ **for** $h = 0{\cdot}2$

(c) $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}, y(0) = 1,$ **for** $x = 0{\cdot}2$ **and** $0{\cdot}4$ **(Gwalior 2017)**

(d) $\frac{dy}{dx} = -xy, y(0) = 1,$ **for** $x = 0{\cdot}2$. **(Rewa 2011)**

While its unique solution is $y = e^{-x^2/2}$.

Solution : (a) $f(x, y) = x + y, x_0 = 0, y_0 = 1, h = 0{\cdot}1$

$\therefore \quad k_1 = h\, f(x_0, y_0) = 0{\cdot}1[0 + 1] = 0{\cdot}1$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0{\cdot}1\left[0 + \frac{0{\cdot}1}{2}, 1 + \frac{0{\cdot}1}{2}\right] = 0{\cdot}11$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0{\cdot}1[0{\cdot}05 + 1{\cdot}055] \doteq 0{\cdot}1105$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0{\cdot}1[0{\cdot}1 + 1{\cdot}1105] = 0{\cdot}12105$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

$$y(0{\cdot}1) = 1 + \frac{1}{6}[0{\cdot}1 + 2(0{\cdot}11 + 0{\cdot}1105) + 0{\cdot}12105] = 1{\cdot}11034.$$

Here at $x = 0{\cdot}2$, $y(0{\cdot}2)$ we will find y,

$x_1 = 0{\cdot}1, \; y_1 = 1{\cdot}11034, h = 0{\cdot}1,$

$$k_1 = hf(x_1, y_1) = 0{\cdot}1[0{\cdot}1, 1{\cdot}11034] = 0{\cdot}121034$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0{\cdot}1\left[0{\cdot}15 + \left(1{\cdot}11034 + \frac{0{\cdot}121034}{2}\right)\right] = 0{\cdot}13208$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0{\cdot}1\left[0{\cdot}15 + \left(1{\cdot}11034 + \frac{0{\cdot}13208}{2}\right)\right] = 0{\cdot}13263$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0{\cdot}1[(0{\cdot}1 + 0{\cdot}1) + (1{\cdot}11034 + 0{\cdot}13263)] = 0{\cdot}14429$$

$$\therefore \quad y_2 = y(0{\cdot}2) = y_1 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

$$= 1{\cdot}11034 + \frac{1}{6}[0{\cdot}121034 + 0{\cdot}26416 + 0{\cdot}26526 + 0{\cdot}14429]$$

$$= 1{\cdot}11034 + \frac{1}{6}[0{\cdot}7948] = 1{\cdot}11034 + 0{\cdot}1325 = 1{\cdot}2428$$

Similarly $x = 0{\cdot}3$, we will find y,

$x_2 = 0{\cdot}2, y_2 = 1{\cdot}2428, h = 0{\cdot}1$

$$k_1 = h\, f(x_2, y_2) = h(x_2 + y_2) = 0{\cdot}1(0{\cdot}2 + 1{\cdot}2428) = 0{\cdot}14428$$

$$k_2 = h\, f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0{\cdot}1\left[0{\cdot}25 + \left(1{\cdot}2428 + \frac{0{\cdot}14428}{2}\right)\right]$$

$$= 0{\cdot}15649$$

$$k_3 = h\, f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0{\cdot}1\left[0{\cdot}25 + (1{\cdot}2428 + \frac{0{\cdot}15649}{2})\right]$$

$$= 0{\cdot}15710$$

$$k_4 = h\, f(x_2 + h, y_2 + k_3) = 0{\cdot}1[0{\cdot}3 + (1{\cdot}2428 + 0{\cdot}15710)]$$

$$= 0{\cdot}16999$$

and $y_3 = y(0\cdot3) = y_2 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 1\cdot2428 + \frac{1}{6}[0\cdot14428 + 0\cdot31298 + 0\cdot31420 + 0\cdot16999]$

$= 1\cdot2428 + \frac{1}{6}(0\cdot94145) = 1\cdot2428 + 0\cdot15690$

$= 1\cdot3997$

To find value of $y_4 = y(0\cdot4)$ at $x = 0\cdot4$

Put $x_3 = 0\cdot3, y_3 = 1\cdot3997, h = 0\cdot1,$

$k_1 = hf(x_3, y_3) = 0\cdot1(0\cdot3 + 1\cdot3997) = 0\cdot16997$

$k_2 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1}{2}\right) = (0\cdot1)\{0\cdot35 + (1\cdot3997 + 0\cdot084985)\} = 0\cdot18347$

$k_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2}{2}\right) = (0\cdot1)\{0\cdot35 + (1\cdot3997 + 0\cdot091735)\} = 0\cdot18414$

$k_4 = hf(x_3 + h, y_3 + k_3) = (0\cdot1)\{0\cdot4 + (1\cdot3997 + 0\cdot18414)\} = 0\cdot19838$

$\therefore\ y_4 = y_3 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 1\cdot3997 + \frac{1}{6}(0\cdot16997 + 0\cdot36694 + 0\cdot36828 + 0\cdot19838)$

$= 1\cdot3997 + \frac{1}{6}(1\cdot10357) = 1\cdot3997 + 0\cdot183928 = 1\cdot5836.$ **Ans.**

These value can be tabulated as :

x	0	0·1	0·2	0·3	0·4
y	1	1·11034	1·2428	1·3997	1·5836

(b) $f(x,y) = 1 + y^2, x_0 = 0, y_0 = 0, h = 0\cdot2$

For the first interval, we have $y_1 = y(0\cdot2)$

$k_1 = hf(x_0, y_0) = (0\cdot2) f(x_0, y_0) = (0\cdot2)(1 + y_0^2) = 0\cdot2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0\cdot2)\left[1 + \left(0 + \frac{0\cdot2}{2}\right)^2\right] = 0\cdot2[1\cdot01] = 0\cdot202$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0\cdot2)\left[1 + \left(0 + \frac{0\cdot202}{2}\right)^2\right] = 0\cdot2[1 + 0\cdot010201]$

$= 0\cdot20204$

$k_4 = hf(x_0 + y_0 + k_3) = 0\cdot2[1 + (0 + 0\cdot20204)^2]$

$= 0\cdot2[1 + 0\cdot04082] = 0\cdot2(1\cdot04082) = 0\cdot208164$

$\therefore\ x = 0\cdot2$, value of y

$y_1 = y(0\cdot2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0 + \frac{1}{6}(0\cdot2 + 0\cdot404 + 0\cdot40408 + 0\cdot208164)$

$= 0\cdot20271$

Now, we shall find y for $x_2 = x_0 + 2h = 0\cdot4$

In this case $x_1 = 0\cdot2, y_1 = 0\cdot20271, h = 0\cdot2$

$k_1 = hf(x_1, y_1) = (0\cdot2)(1 + y_1^2) = 0\cdot2\left[1 + (0\cdot20271)^2\right]$

$= 0\cdot20822$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0\cdot2)\left[1 + \left(0\cdot20271 + \frac{0\cdot20822}{2}\right)^2\right]$

$= (0\cdot2)\left[1 + (0\cdot30682)^2\right] = 0\cdot21883$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0\cdot2)\left[1 + \left(0\cdot20271 + \frac{0\cdot21883}{2}\right)^2\right]$

$= (0\cdot2)\left[1 + (0\cdot31213)^2\right] = 0\cdot21948$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0\cdot2)\left[1 + (0\cdot42219)^2\right] = 0\cdot23565$

$\therefore y_2 = y(0\cdot4) = y_1 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$= 0\cdot20271 + \frac{1}{6}[0\cdot20882 + 0\cdot43766 + 0\cdot43896 + 0\cdot23565]$

$= 0\cdot42279$

Now, we shall find y for $x_3 = x_0 + 3h = 0\cdot6$

In this case $x_2 = 0\cdot4, y_2 = 0\cdot42279, h = 0\cdot2$

$k_1 = hf(x_2, y_2) = (0\cdot2)\left[1 + (0\cdot42279)^2\right] = 0\cdot23575$

$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0\cdot2)\left[1 + \left(0\cdot42279 + \frac{0\cdot23575}{2}\right)^2\right]$

$= (0\cdot2)\left[1 + (0\cdot54067)^2\right] = 0\cdot25846$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0\cdot2)\left[1 + \left(0\cdot42279 + \frac{0\cdot25846}{2}\right)^2\right]$

$= (0\cdot2)\left[1 + (0\cdot55202)^2\right] = 0\cdot26095$

$$k_4 = hf(x_2+h, y_2+k_3) = (0\cdot2)\left[1+(0\cdot42279+0\cdot26095)^2\right]$$

$$= 0\cdot29350$$

$$\therefore\ y_3 = y(0\cdot6) = y_2 + \frac{1}{6}(s_1+2s_2+2s_3+s_4)$$

$$= 0\cdot42279 + \frac{1}{6}(0\cdot23575+0\cdot51692+0\cdot52190+0\cdot29350)$$

$$= 0\cdot68413.$$ **Ans.**

The value of x and y are tabulated as :

x	0	0 ·2	0 ·4	0 ·6
y	0	0 · 20271	0 · 42279	0 · 68413

(c) $f(x,y) = \dfrac{y^2-x^2}{y^2+x^2}, x_0 = 0, y_0 = 1, h = 0\cdot2$

$$k_1 = hf(x_0,y_0) = (0\cdot2)\frac{y_0^2-x_0^2}{y_0^2+x_0^2} = (0\cdot2)\left(\frac{1^2-0^2}{1^2+0^2}\right) = 0\cdot2$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0\cdot2)\left[\frac{\left(y_0+\frac{0\cdot2}{2}\right)^2-\left(x_0+\frac{0\cdot2}{2}\right)^2}{\left(y_0+\frac{0\cdot2}{2}\right)^2+\left(x_0+\frac{0\cdot2}{2}\right)^2}\right]$$

$$= (0\cdot2)\left[\frac{(1\cdot1)^2-(0\cdot1)^2}{(1\cdot1)^2+(0\cdot1)^2}\right] = (0\cdot2)\times\frac{1\cdot2}{1\cdot22} = 0\cdot19671$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0\cdot2)\left[\frac{(1\cdot09836)^2-(0\cdot1)^2}{(1\cdot09836)^2+(0\cdot1)^2}\right]$$

$$= 0\cdot19671$$

$$k_4 = hf(x_0+h, y_0+k_3) = (0\cdot2)f(0\cdot2, 1\cdot19671)$$

$$= (0\cdot2)\left[\frac{(1\cdot19671)^2-(0\cdot2)^2}{(1\cdot19671)^2+(0\cdot2)^2}\right] = 0\cdot18913$$

$$\therefore\ y_1 = y(0\cdot2) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$$

$$= 1+\frac{1}{6}[0\cdot2+0\cdot39342+0\cdot39342+0\cdot18913]$$

$$= 1\cdot19600$$

To find the value of y at $x = 0\cdot4$. In this case $x_1 = 0\cdot2, y_1 = 1\cdot196,\ h = 0\cdot2$

$$k_1 = hf(x_1,y_1) = h\left(\frac{y_1^2-x_2^2}{y_1^2+x_2^2}\right) = (0\cdot2)\left[\frac{(1\cdot196)^2-(0\cdot2)^2}{(1\cdot196)^2+(0\cdot2)^2}\right] = 0\cdot18912$$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = (0\cdot2)[0\cdot3, 1\cdot29056] = 0\cdot17949$$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = (0\cdot2)[0\cdot3, 1\cdot28575] = 0\cdot17935$$

$$k_4 = hf(x_1+h, y_1+k_3) = (0\cdot2)[0\cdot4, 1\cdot37535] = 0\cdot16880$$

$$\therefore\ y_2 = y(0\cdot4) = y_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 1\cdot196+\frac{1}{6}[0\cdot18912+0\cdot35898+0\cdot35870+0\cdot16880]$$

$$= 1\cdot37527.$$ **Ans.**

(d) $f(x,y) = -xy, x_0 = 0, y_0 = 1, h = 0\cdot2$

$$k_1 = hf(x_0,y_0) = (0\cdot2)f(0,1) = 0$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0\cdot2)f(0\cdot1,1) = -0\cdot02$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0\cdot2)f(0\cdot1, 0\cdot99) = -0\cdot0198$$

$$k_4 = hf(x_0+h, y_0+k_3) = (0\cdot2)f(0\cdot2, 1-0\cdot0198)$$

$$= (0\cdot2)f(0\cdot2, 0\cdot9802) = -(0\cdot2)(0\cdot2)(0\cdot9802) = -0\cdot039208$$

$$\therefore\ y_1 = y(x_0+h) = y(0\cdot2) = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= 1+\frac{1}{6}[0+2(-0\cdot02)+2(-0\cdot0198)+(-0\cdot039208)]$$

$$= 1+\frac{1}{6}[0-0\cdot04-0\cdot0396-0\cdot039208]$$

$$= 1-\frac{1}{6}(0\cdot118808) = 1-0\cdot019801 = 0\cdot9801986$$

$$= 0\cdot9802$$ (up to four place of decimal value)

But given that the unique solution of the given problem is $y = e^{-x^2/2}$.

$\therefore$ at $x = 0\cdot2$ the value of y is

$$y(0\cdot2) = e^{-(0\cdot2)^2/2} = e^{-0\cdot02} = 0\cdot98011986 \approx 0\cdot98012.$$ **Ans.**

Q. 18. Find by Runge-kutta method an approximation value of y for $x = 0{\cdot}1$, given that $y(0) = 1$ and $\frac{dy}{dx} = 3x + y^2$.

(Indore 2011, 14, 16; Ujjain 11, 17; Gwalior 16; Bhopal 22)

Solution : Here $x_0 = 0, y_0 = 1, h = 0{\cdot}1, f(x, y) = 3x + y^2$

$$k_1 = hf(x_0, y_0) = h[3x_0 + y_0^2] = 0{\cdot}1[3 \times 0 + 1^2] = 0{\cdot}1(1) = 0{\cdot}1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= h\left[3\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_1}{2}\right)^2\right] = 0{\cdot}1\left[3\left(0 + \frac{0{\cdot}1}{2}\right) + \left(1 + \frac{0{\cdot}1}{2}\right)^2\right]$$

$$= 0{\cdot}1\left[\frac{0{\cdot}3}{2} + (1 + 0{\cdot}05)^2\right]$$

$$= 0{\cdot}1[0{\cdot}15 + (1{\cdot}05)^2] = 0{\cdot}12525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\left[3\left(x_0 + \frac{h}{2}\right) + \left(y_0 + \frac{k_2}{2}\right)^2\right]$$

$$= 0{\cdot}1\left[3\left(0 + \frac{0{\cdot}1}{2}\right) + \left(1 + \frac{0{\cdot}12525}{2}\right)^2\right] = 0{\cdot}127917$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0{\cdot}1[3(x_0 + h) + (y_0 + k_3)^2]$$

$$= 0{\cdot}1[3(0 + 0{\cdot}1) + (1 + 0{\cdot}127917)^2]$$

$$= 0{\cdot}1[0{\cdot}3 + (1{\cdot}127917)^2]$$

$$= 0{\cdot}15721967$$

$$y(0{\cdot}1) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}[0{\cdot}1 + 2(0{\cdot}12525) + 2(0{\cdot}127917) + 0{\cdot}15721]$$

$$= 1 + \frac{1}{6}(0{\cdot}763544) = 1 + 0{\cdot}127257 = 1{\cdot}127257.$$ **Ans.**

Q. 19. Apply Runge's method to approximate y where $x = 1{\cdot}6$ given that $\frac{dy}{dx} = x - y$, $y(1) = 0{\cdot}4$.

(Jabalpur 2011; Ujjain 11, 15; Sagar 14)

Solution : Here $x_0 = 1, y_0 = 0{\cdot}4, h = 0{\cdot}6, f(x, y) = x - y$

$$\Rightarrow \quad k_1 = hf[x_0, y_0] = 0{\cdot}6[x_0 - y_0] = 0{\cdot}6[1 - 0{\cdot}4] = 0{\cdot}36$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0{\cdot}6)\left[\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0{\cdot}6)\left[\left(1 + \frac{0{\cdot}6}{2}\right) - \left(0{\cdot}4 + \frac{0{\cdot}36}{2}\right)\right]$$

$$= 0{\cdot}432.$$

$$k' = hf(x_0 + h, y_0 + k_1) = (0{\cdot}6)[(x_0 + h) - (y_0 + k_1)]$$

$$= (0{\cdot}6)[(1 + 1{\cdot}6) - (0{\cdot}4 + 0{\cdot}36)]$$

$$= 0{\cdot}504$$

$$k_3 = hf(x_0 + h, y_0 + k') = (0{\cdot}6)[(x_0 + h) - (y_0 + k')]$$

$$= 0{\cdot}6[(1 + 0{\cdot}6) - (0{\cdot}4 + 0{\cdot}504)] = 0{\cdot}4176$$

Hence $k = \frac{1}{6}[k_1 + 4k_2 + k_3] = \frac{1}{6}[0{\cdot}36 + 4(0{\cdot}432) + 0{\cdot}4176]$

$$= 0{\cdot}4176$$

Hence approximate value of $y = y_0 + k$ is

$$= 0{\cdot}4 + 0{\cdot}4176$$

$$= 0{\cdot}8176.$$ **Ans.**

Q. 20. Apply Milne's method to find a solution of the differential equation $\frac{dy}{dx} = x - y^2$ in the range $0 \le x \le 1$, for the boundary condition $y = 0$ at $x = 0$.

(Bhopal 2013; Gwalior 2017)

Solution : Here $y' = f(x, y) = x - y^2, x_0 = 0, y_0 = 0$

Taking $h = 0{\cdot}2$, so that

$x_1 = x_0 + h = 0{\cdot}2, x_2 = x_0 + 2h = 0{\cdot}4, x_3 = 0{\cdot}6, x_4 = 0{\cdot}8, x_5 = 1{\cdot}0$

Firstly we shall compute y_1, y_2, y_3 using Picard's method :

$$y^{(n)} = y_0 + \int_0^x f(x, y^{(n-1)})\,dx$$

$$y^{(1)} = 0 + \int_0^x f(x, y^{(0)})\,dx = \int_0^x f(x, 0)\,dx = \int_0^x x\,dx = \frac{x^2}{2}$$

$$[\because f(x, y) = x - y^2, y(0) = 0]$$

$$y^{(2)} = 0 + \int_0^x f(x, y^{(1)})\,dx = \int_0^x \left[x - \left(\frac{x^2}{2}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20}$$

$$y^{(3)} = 0 + \int_0^3 f(x, y^{(2)})\,dx = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx$$

$$= \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}$$

Taking $h = 0\cdot2$,

$$y_1 = y(0\cdot2) = \left(\frac{x^2}{2}\right)_{(0\cdot2)} = 0\cdot02$$

$$y_2 = y(0\cdot4) = \left[\frac{x^2}{2} - \frac{x^5}{20}\right]_{(0\cdot4)} = 0\cdot079488 \approx 0\cdot0795$$

$$y_3 = y(0\cdot6) = \left[\frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}\right]_{(0\cdot6)} = 0\cdot176112 \approx 0\cdot176$$

$$y'_0 = (y')_{x=0} = [f(x,y)]_{x=0} = x_0 - y_0^2 = 0$$

$$y'_1 = f(x_1,y_1) = f_1 = x_1 - y_1^2 = 0\cdot2 - (0\cdot02)^2 = 0\cdot2 - 0\cdot0004 = 0\cdot1996$$

$$y'_2 = f(x_2,y_2) = f_2 = x_2 - y_2^2 = 0\cdot4 - (0\cdot0795)^2 = 0\cdot3937$$

$$y'_3 = f(x_3,y_3) = f_3 = x_3 - y_3^2 = 0\cdot6 - (0\cdot176)^2 = 0\cdot5690$$

By Milne's predictor formula,

$$y_4 = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] = 0 + \frac{4(0\cdot2)}{3}[0\cdot3992 - 0\cdot3937 + 1\cdot1380]$$

$$= 0\cdot3049$$

and $y'_4 = f_4 = x_4 - y_4^2 = 0\cdot8 - (0\cdot3049)^2 = 0\cdot7070$

Using Milne's corrector formula,

$$y_4^{(1)} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$= 0\cdot0795 + \frac{0\cdot2}{3}(0\cdot3937 + 2\cdot2760 + 0\cdot7070) = 0\cdot3046$$

Thus $y'_4 = f_4 = 0\cdot8 - (0\cdot3046)^2 = 0\cdot7072$

$\therefore$ Again Milne's predictor formula,

$$y_5 = y_1 + \frac{4h}{3}[2y'_2 - y'_3 + 2y'_4]$$

$$= 0\cdot02 + \frac{4\times(0\cdot2)}{3}[2\times0\cdot3937 - 0\cdot5690 + 2\times0\cdot7072]$$

$$= 0\cdot455413 \approx 0\cdot4554$$

$$y'_5 = f_5 = f(x_5,y_5) = x_5 - y_5^2 = 1\cdot0 - (0\cdot4554)^2 = 0\cdot7926$$

Using corrector formula,

$$y_5^{(1)} = y_3 + \frac{h}{3}\left[y'_3 + 4y'_4 + y'_5\right]$$

$$= 0\cdot176 + \frac{0\cdot2}{3}[0\cdot5690 + 4\times0\cdot7072 + 0\cdot7926]$$

$$= 0\cdot45536 \approx 0\cdot4554.$$

Ans.

Tabulated table is :

x	0	0·2	0·4	0·6	0·8	1·0
y	0	0·02	0·0795	0·176	0·3046	0·4554

Q. 21. Explain Milne-Simpson method. **(Gwalior 2022)**

Solution : Milne-Simpson formula

$$\frac{dy}{dx} = f(x,y), \quad y = y_0, \text{ where } x = x_0 \qquad ...(1)$$

$f(x,y)$ is replaced by Newton's forward interpolation formula

$$\frac{dy}{dx} = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0 ..... \qquad ...(2)$$

Where $p = \dfrac{x - x_0}{h}$

Integrating (2) over the interval x to x_4, we obtain

$$\int_{x_0}^{x_4} y\,dx = \int_{x_0}^{x_4}\left[f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0\right.$$

$$\left. + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f_0\right]dx$$

Since $p = \dfrac{x - x_0}{h}$, the limits of integration become $p = 0$ to $p = 4$ and $dx = hdp$

$$y_4 - y_0 = h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \frac{14}{45}\Delta^4 f_0\right]$$

Replacing $\Delta f_0, \Delta^2 f_0$ and $\Delta^3 f_0$ by their functional values, we obtain

$$y_4 = y_0 + h\left[4f_0 + 8(f_1 - f_0) + \frac{20}{3}(f_2 - 2f_1 + f_0)\right.$$

$$\left. + \frac{8}{3}(f_3 - 3f_2 + 3f_1 - f_0)\right] + \frac{14}{45}h\Delta^4 f_0$$

$$= y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) + \frac{14}{45}h\Delta^4 f_0$$

Neglecting fourth and higher order difference, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \quad \ldots(3)$$

More generally equation (2) can be expressed as

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2f_n - f_{n-1} + 2f_{n-2}) \quad \ldots(4)$$

Which is known as predictor formula or Milne's formula.

Q. 22. Using Milne's method, solve $\frac{dy}{dx} = 2e^x - y$ for $x = 0{\cdot}4$ in the terms of 0·1

x	0	0·1	0·2	0·3
y	2·00	2·01	0·04	0·09

(Jabalpur 2011; Indore 12)

Solution : $f(x,y) = 2e^x - y$ and $h = 0{\cdot}1$

To find y_4 for $x_4 = x_0 + 4h = 0 + 4(0{\cdot}1) = 0{\cdot}4$

Predictor formula $y_4 = y_0 + \frac{4}{3}h(2y_1' - y_2' + 2'y_3')$ ...(1)

$$y_1' = f(x_1, y_1) = 2e^{x_1} - y_1 = 2.e^{0{\cdot}1} - 2{\cdot}01 = 2'(1{\cdot}1051) - 2{\cdot}01$$
$$= 0{\cdot}2003$$

$$y_2' = f(x_2, y_2) = 2e^{x_2} - y_2 = 2.e^{0{\cdot}2} - 0{\cdot}04 = 2(1{\cdot}2213) - 0{\cdot}04$$
$$= 2{\cdot}4427$$

$$y_3' = f(x_3, y_3) = 2e^{x_3} - y_3 = 2.e^{0{\cdot}3} - 0{\cdot}09 = 2(1{\cdot}3498) - 0{\cdot}09$$
$$= 2{\cdot}6096$$

Predicted value

$$y_4 = 2 + \frac{4}{3}(0{\cdot}1)[2 \times 0{\cdot}2003 - 2{\cdot}4427 + 2 \times 2{\cdot}6096]$$

$$= 2 + \frac{0{\cdot}4}{3}[3{\cdot}1771] = 2{\cdot}4236$$

Now, $x_4 = 0{\cdot}4, y_4 = 2{\cdot}4236$

$$y_4' = f(x_4, y_4) = 2e^{x_4} - y_4 = 2.e^{0{\cdot}4} - 2{\cdot}4236$$
$$= 2(1{\cdot}4917) - 2{\cdot}4236$$
$$= 0{\cdot}5599$$

Put the value in corrector formula $y_4 = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$, we get

$$= 0{\cdot}04 + \frac{0{\cdot}1}{3}(2{\cdot}4427 + 4 \times 2{\cdot}6096 + 0{\cdot}5599)$$
$$= 0{\cdot}04 + \frac{0{\cdot}1}{3}(13{\cdot}441)$$
$$= 0{\cdot}4880.$$

Ans.

Q. 23. Solve by Milne's method :

(a) If $y(x)$ is a solution of the equation $\frac{dy}{dx} = \frac{1}{2}(x + y)$ then find the value of $y(2)$, given that : **(Bhopal 2014; Ujjain 14, 17; Gwalior 16)**

x	0	0·5	1·0	1·5
y	2	2·636	3·595	4·968

(b) Find $y(1)$, given for $\frac{dy}{dx} = 2 - xy^2$, $y(0) = 1{\cdot}0$ and $h = 0{\cdot}2$.

(c) The range of $\frac{dy}{dx} = x - y^2, y(0) = 0$ is $0 \le x \le 1$. **(Gwalior 2011)**

(d) $\frac{dy}{dx} = x + y^2$, $y(0) = 1$ **(Gwalior 2012)**

$h = 0{\cdot}1$ and take accuracy of 0·000001 and find y at $x = 0{\cdot}4$.

Solution : (a) $f(x,y) = \frac{1}{2}(x + y)$ and $h = 0{\cdot}5$ ...(1)

For $x_4 = x_0 + 4h = 0 + 4 \times (0{\cdot}5) = 2$ we have to find value of y_4.

By Milne's predictor formula,

$$y_4 = y_0 + \frac{4}{3}h(2y_1' - y_2' + 2y_3') \quad \ldots(2)$$

$$y_1' = f(x_1, y_1) = \frac{1}{2}(0{\cdot}5 + 2{\cdot}636) = 1{\cdot}568$$

$$y_2' = f(x_2, y_2) = \frac{1}{2}(1{\cdot}0 + 3{\cdot}595) = 2{\cdot}2975$$

$$y_3' = f(x_3, y_3) = \frac{1}{2}(1{\cdot}5 + 4{\cdot}968) = 3{\cdot}234$$

Put the above values in eqn. (2), we have

$$y_4 = 2 + \frac{4}{3} \times (0{\cdot}5)[2 \times 1{\cdot}568 - 2{\cdot}2975 + 2 \times 3{\cdot}234]$$
$$= 2 + 0{\cdot}666[3{\cdot}136 - 2{\cdot}297 + 6{\cdot}468] = 6{\cdot}8713$$

Now, put $x_4 = 2$ and $y_4 = 6{\cdot}8713$,

$$y_4' = \frac{1}{2}(2 + 6{\cdot}8713) = 4{\cdot}4357$$

Now again applying corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

Put the value of y_2', y_3', y_4' in above formula

$$y_4 = 3\cdot595+\frac{0\cdot5}{3}(2\cdot2975+4\times3\cdot234+4\cdot4357)$$

$$= 3\cdot595+3\cdot2782 = 6\cdot8732$$

Now, $y_4' = \frac{1}{2}(2+6\cdot8732) = 4\cdot4366$

Taken from corrector formula, we get

$$y_4 = 3\cdot595+\frac{0\cdot5}{3}(2\cdot2975+4\times3\cdot234+4\cdot4366)$$

$$= 6\cdot8733.$$ **Ans.**

(b) Here, $f(x,y) = 2-xy^2, y(0) = 1, h = 0\cdot2$

Now we shall obtain the three consecutive values of y and y' corresponding to $x = 0\cdot2$, 0·4 and 0·6 by ($\because h = 0\cdot2$) Picard method

According to this

$$y = y_0 + \int_0^x f(x,y)dx$$

For first approx. $y = 1$,

$$y^{(1)} = 1\cdot0+\int_0^x f(x,1)dx = 1+\int_0^x(2-x)dx = 1+2x-\frac{x^2}{2} \quad ...(1)$$

Second approx.,

$$y^{(2)} = y_0\int_0^x f(x,y^{(1)})dx = 1+\int_0^x\{2-x(1+2x-\frac{x^2}{2})^2\}dx$$

$$= 1+2x-\frac{x^2}{2}-\frac{4x^3}{2}-\frac{3x^4}{4}+\frac{2x^5}{5}-\frac{1}{24}x^6 \quad ...(2)$$

Third approx.,

$$y^{(3)} = y_0+\int_0^x f(x,y^2)dx$$

$$= 1+2x-\frac{x^2}{2}-\frac{4x^3}{3}-\frac{3x^4}{4}+\frac{14}{15}x^5+\frac{79}{72}x^6+... \quad ...(3)$$

Now, put values of x, 0·2, 0·4, 0·6 in eqns. (1), (2) and (5) respectively we have,

$y_1 = y(0\cdot2) = 1\cdot3685$

$y_2 = y(0\cdot4) = 1\cdot6298$

$y_3 = y(0\cdot6) = 1\cdot7621$

Now, by Milne's method, we shall obtain the values y_4 and y_5 of y corresponding to $x = 0\cdot8$ and 1·0, by Milne's predictor formula

$$y_4 = y_0+\frac{4}{3}h[2y_1'-y_2'+2y_3'] \quad ...(4)$$

Now, $y_1' = [f(x,y)]_{x=x_1} = f(x_1,y_1) = 2-x_1y_1^2 = 2-(0\cdot2)(1\cdot3685)^2$

$= 2-0\cdot3746 = 1\cdot6254$

Again, put $x = 0\cdot4, y_2 = 1\cdot6298$,

$y_2' = f(x_2, y_2') = 2-x_2y_2^2 = 2-(0\cdot4)(1\cdot6298)^2 = 0\cdot9375$

Similarly,

$y_3' = 2-x_3y_3^2 = 2-(0\cdot6)(1\cdot7621)^2 = 0\cdot1370$

Put in eqn. (4),

$$y_4 = 1+\frac{4}{3}(0\cdot2)[2\times1\cdot6254-0\cdot9375+2\times0\cdot1370] = 1\cdot6899$$

By corrected formula,

$$y_{n+1}^{(1)} = y_{n-3}+\frac{h}{3}(y'_{n-1}+4y'_n+y'_{n+1})$$

Put $x = 0\cdot8, y_4 = 1\cdot6899$,

$$y_4^{(1)} = y_2+\frac{h}{3}(y'_2+4y'_3+y'_4) \text{ where } y_4' = 2-x_4y_4^2$$

$$= y_2+\frac{h}{3}\left[y'_2+4y'_3+\left\{2-(0\cdot8)(1\cdot6899)^2\right\}\right]$$

$$= 1\cdot6298+\frac{0\cdot2}{3}[0\cdot9375+4(0\cdot1370)+(-0\cdot2846)] = 1\cdot7098$$

Now put, $y'_4 = 2-x_4y_4^2$, $x_4 = 0\cdot8, y_4 = 1\cdot7098$,

$y_4' = 2-(0\cdot8)(1\cdot7098)^2 = -0\cdot3387$

By predictor formula,

$$y_5 = y_1+\frac{4h}{3}[2y'_2-y'_3+2y'_4]$$

$$= 1\cdot3685+\frac{4(0\cdot2)}{3}[2\times0\cdot9375-0\cdot1370+2(-0\cdot3387)]$$

$$= 1\cdot6513$$

and $y_5' = 2-x_5y_5^2 = 2-(1\cdot0)(1\cdot6513)^2 = -0\cdot7267$

Correct value of y_5

$$y'_5 = y_3+\frac{h}{3}\left[y'_3+4y'_4+y'_5\right]$$

$$= 1\cdot7621+\frac{0\cdot2}{3}[0\cdot1370+4(-0\cdot3387)+(-0\cdot7267)]$$

$$= 1\cdot7621+\frac{0\cdot2}{3}(-1\cdot9455) = 1\cdot7621-0\cdot1296$$

$$= 1\cdot6325 = y(1).$$ **Ans.**

(c) To begin with y_1, y_2, y_3 is required, to find which on order four can be used by any single method. Picards method is convenient according to this,

$$y^{(n)} = y_0 + \int_0^x f(x, y^{(n-1)})dx$$

$$y^{(1)} = 0 + \int_0^x f(x, y^{(0)})dx = \int_0^x f(x,0)dx = \int_0^x x\,dx = \frac{x^2}{2}$$

Because $f(x,y) = x - y^2$, $y(0) = 0$

$$y^{(2)} = 0 + \int_0^x f(x, y^{(1)})dx = \int_0^x \left[x - \left(\frac{x^2}{2}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20}$$

$$y^{(3)} = 0 + \int_0^3 f(x, y^{(2)})dx = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}$$

Taking $h = 0{\cdot}2$,

$$y_1 = y(0{\cdot}2) = \left(\frac{x^2}{2}\right)_{(0\text{-}2)} = 0{\cdot}02$$

$$y_2 = y(0{\cdot}4) = \left[\frac{x^2}{2} - \frac{x^5}{20}\right]_{(0\text{-}4)} = 0{\cdot}079488 \approx 0{\cdot}0795$$

$$y_3 = y(0{\cdot}6) = \left[\frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}\right]_{(0\text{-}6)} = 0{\cdot}176112 \approx 0{\cdot}176$$

$$y'_0 = (y')_{x=0} = [f(x,y)]_{x=0} = x_0 - y_0^2 = 0$$

$$y'_1 = f(x_1, y_1) = f_1 = x_1 - y_1^2 = 0{\cdot}2 - (0{\cdot}02)^2 = 0{\cdot}2 - 0{\cdot}0004 = 0{\cdot}1996$$

$$y'_2 = f(x_2, y_2) = f_2 = x_2 - y_2^2 = 0{\cdot}4 - (0{\cdot}0795)^2 = 0{\cdot}3937$$

$$y'_3 = f(x_3, y_3) = f_3 = x_3 - y_3^2 = 0{\cdot}6 - (0{\cdot}176)^2 = 0{\cdot}5690$$

From the predictor formula,

$$y_4 = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right] = 0 + \frac{4(0{\cdot}2)}{3}[0{\cdot}3992 - 0{\cdot}3937 + 1{\cdot}1380]$$

$$= 0{\cdot}3049$$

and $y'_4 = f_4 = x_4 - y_4^2 = 0{\cdot}8 - (0{\cdot}3049)^2 = 0{\cdot}7070$

From the corrector formula,

$$y_4^{(1)} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$= 0{\cdot}0795 + \frac{0{\cdot}2}{3}(0{\cdot}3937 + 2{\cdot}2760 + 0{\cdot}7070) = 0{\cdot}3046$$

Now, put $y'_4 = f_4 = 0{\cdot}8 - (0{\cdot}3046)^2 = 0{\cdot}7072$

So, from the predictor formula,

$$y_5 = y_1 + \frac{4h}{3}[2y'_2 - y'_3 + 2y'_4]$$

$$= 0{\cdot}02 + \frac{4 \times (0{\cdot}2)}{3}[2 \times 0{\cdot}3937 - 0{\cdot}5690 + 2 \times 0{\cdot}7072]$$

$$= 0{\cdot}455413 \approx 0{\cdot}4554$$

$$y'_5 = f_5 = f(x_5, y_5) = x_5 - y_5^2 = 1{\cdot}0 - (0{\cdot}4554)^2 = 0{\cdot}7926$$

From the corrector formula,

$$y_5^{(1)} = y_3 + \frac{h}{3}\left[y'_3 + 4y'_4 + y'_5\right]$$

$$= 0{\cdot}176 + \frac{0{\cdot}2}{3}[0{\cdot}5690 + 4 \times 0{\cdot}7072 + 0{\cdot}7926]$$

$$= 0{\cdot}45536 \approx 0{\cdot}4554.$$

Tabulated Table is :

x	0	0·2	0·4	0·6	0·8	1·0
y	0	0·02	0·0795	0·176	0·3046	0·4554

Ans.

(d) Here $f(x,y) = x + y^2$, $x_0 = 0, y_0 = 1$, $h = 0{\cdot}1$.

By single step method and then

We shall find $y'_1 = f_1, y'_2 = f_2, y'_3 = f_3$ calculate y_4 by Euler's method

$$y_m = y_{m-1} + hf(x_{m-1}, y_{m-1})$$

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0^2) = 1 + 0{\cdot}1(0 + (1)^2) = 1{\cdot}1$$

$$y_2 = y_1 + hf(x_1, y_1) = 1{\cdot}1 + (0{\cdot}1)[(0{\cdot}1) + (1{\cdot}1)^2] = 1{\cdot}231.$$

$$y_3 = y_2 + hf(x_2, y_2) = 1{\cdot}231 + (0{\cdot}1)[(0{\cdot}2) + (1{\cdot}231)^2] = 1{\cdot}4025361.$$

and $y'_1 = f_1 = 1{\cdot}31$,

$$y'_2 = f_2 = 1{\cdot}715361,$$

$$y'_3 = f_3 = 2{\cdot}2671075.$$

$\therefore$ Predictor formula,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 1{\cdot}0 + \frac{4(0{\cdot}1)}{3}[2(1{\cdot}31) - 1{\cdot}715361 + 2(2{\cdot}2671075)] = 1{\cdot}7251805$$

$$y_4' = f_4^{(p)} = f(x_4, y_4^{(p)}) = f(0\cdot4, 1\cdot7251805)$$

$$= 0\cdot4 + (1\cdot7251805)^2 = 3\cdot3762479$$

Corrector formula,

$$y_4^{(c_1)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1\cdot231 + \frac{0\cdot1}{3}(1\cdot715361 + 4\times2\cdot2671075 + 3\cdot3762479) = 1\cdot7030013.$$

Now, put corrected value of y_4, 1·7030013

$$y_4' = f_4^{(c_1)} = f(x_4, y_4^{(c_1)}) = f(0\cdot4, 1\cdot7030013)$$

$$= 0\cdot4 + (1\cdot7030013)^2 = 3\cdot3002134.$$

Again, by corrected formula, second corrected value of y_4

$$y_4^{(2)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= 1\cdot231 + \frac{0\cdot1}{3}(1\cdot715361 + 4\times2\cdot2671075 + 3\cdot3002134)$$

$$= 1\cdot6905308$$

and $$y'_4 = f_4^{(c_2)} = f(x_4, y_4^{(c_2)})$$

$$= x_4 + (y_4^{(c_2)})^2 = 0\cdot4 + (1\cdot6905308)^2 = 3\cdot2578944$$

Similarly, third corrected value of y_4,

$$y_4^{(c_3)} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4')$$

$$= y_2 + \frac{h}{3}(y_2' + 4y_3' + f_4^{(c_2)})$$

$$= y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4^{(c_2)})$$

$$= 1\cdot6990562$$

$$f_4^{(3)} = x_4 + \left(y_4^{(c_3)}\right)^2 = 0\cdot4 + (1\cdot6990562)^2 = 3\cdot286792$$

$$y_4^{(c_4)} = 1\cdot700194$$

$$f_4^{(c_4)} = f(x_4, y_4^{(c_4)}) = 3\cdot2900661$$

$$y_4^{(c_5)} = 1\cdot7001286, f_4^{(c_5)} = f(x_4, y_4^{(c_5)}) = 3\cdot2904371$$

$$y_4^{(c_6)} = 1\cdot7001409, f_4^{(c_6)} = f(x_4, y_4^{(c_6)}) = 3\cdot2904792$$

$$y_4^{(c_7)} = 1\cdot7001423$$

$$|y_4^{(c_7)} - y_4^{(c_6)}| = 0\cdot0000014$$

Which is desirable accuracy, 0·000001

$$|y_4^{(p)} - y_4^{(c_1)}| = |1\cdot7251805 - 1\cdot7030013| = 0\cdot0221792$$

Accuracy is not up to 0·000001, hence process can be continue. **Ans.**

Q. 24. Tabulate by Milne's method the numerical solutions of $\frac{dy}{dx} = x + y$ with initial conditions $x_0 = 0$, $y_0 = 1$ from $x = 0\cdot20$ to $x = 0\cdot30$. **(Rewa 2016)**

Solution : Given equation is

$$\frac{dy}{dx} = x + y \Rightarrow \frac{dy}{dx} - y = x$$

$$\therefore \quad (P.I.) = e^{\int -dx} = e^{-x}$$

Thus the solution is :

$$y.e^{-x} = \int x.e^{-x}dx + c = -xe^{-x} - e^{-x} + c \Rightarrow y = -x - 1 + ce^{x}$$

Initially $x_0 = 0, y_0 = 1 \Rightarrow c = 2$

$$\therefore \quad y = -x - 1 + 2e^{x}$$

Let $h = 0\cdot05$

$$\therefore \quad y = -0\cdot05 - 1 + 2e^{0\cdot05} = 1\cdot0525 \text{ at } x = 0\cdot05$$

and $$y = -0\cdot1 - 1 + 2e^{0\cdot1} = 1\cdot1103 \text{ at } x = 0\cdot10$$

and $$y = -0\cdot15 - 1 + 2e^{0\cdot15} = 1\cdot1737 \text{ at } x = 0\cdot15.$$

Hence, we have

x	y	$y' = \frac{dy}{dx} = x + y$
$x_0 = 0\cdot00$	$y_0 = 1\cdot0000$	$y_0' = 1\cdot0000$
$x_1 = 0\cdot05$	$y_1 = 1\cdot0525$	$y_1' = 1\cdot1025$
$x_2 = 0\cdot10$	$y_2 = 1\cdot1103$	$y_2' = 1\cdot2103$
$x_3 = 0\cdot15$	$y_3 = 1\cdot1736$	$y_3' = 1\cdot3236$

By Milne predictor formula,

$$y_2 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 1 + \frac{4(0\cdot05)}{3}[2\times1\cdot1026 - 1\cdot2103 + 2\times1\cdot3237] = 1\cdot2428$$

$$\therefore \quad y'_4 = x_4 + y_4 = 0\cdot2 + 1\cdot2428 = 1\cdot4428.$$

y_4 is the predicted value of y corresponding to $x = 0{\cdot}20$ which is corrected by

$$y_4^{(1)} = y_2 + \frac{y}{3}[y'_2 + 4y'_3 + 4y'_4]$$

$$= 1{\cdot}1103 + \frac{0{\cdot}05}{3}[1{\cdot}2103 + 4(1{\cdot}3237) + 1{\cdot}4428] = 1{\cdot}2428$$

Which is same as the predicted value.

$y_4 = y_{0{\cdot}20} = 1{\cdot}2428$ and $y'_4 = 1{\cdot}4428$

Now, we are to find the values of y corresponding to $x = 0{\cdot}25$ and $x = 0{\cdot}3$, *i.e.*, y_5 and y_6.

Milne's predictor formula for y_5 is

$$y_5 = y_1 + \frac{4h}{3}[2y'_2 + y'_3 + 2y'_4]$$

$$= 1{\cdot}0525 + \frac{4(0{\cdot}05)}{3}[2(1{\cdot}2103) - 1{\cdot}3237 + 2(1{\cdot}4428] = 1{\cdot}3180$$

Which is corrected by

$$y_5^{(1)} = y_3 + \frac{h}{3}[y'_3 + 4y'_4 + y'_5]$$

$$= 1{\cdot}1737 + \frac{0{\cdot}05}{3}[1{\cdot}3237 + 4(1{\cdot}4428) + 1{\cdot}5680]$$

$(\because y'_2 = x_5 + y_5 = 1{\cdot}5681)$

$$= 1{\cdot}3181$$

Which is same as the predicted value.

$\therefore$ $y_5 = y_{0{\cdot}25} = 1{\cdot}3181$ and $y'_5 = 1{\cdot}5681$

Now, Milne's predictor formula for y_6 is :

$$y_6 = y_2 + \frac{4h}{3}[2y'_3 - y'_4 + 2y'_5]$$

$$= 1{\cdot}1103 + \frac{4(0{\cdot}5)}{3}[2(1{\cdot}3237) - 1{\cdot}4428 + 2(1{\cdot}5681)] = 1{\cdot}3997$$

Which is corrected by

$$y_6^{(1)} = y_4 + \frac{h}{3}[y'_4 + 4y'_5 + y'_6]$$

$$= 1{\cdot}2428 + \frac{0{\cdot}05}{4}[1{\cdot}4428 + 4(1{\cdot}5681) + 1{\cdot}6997] = 1{\cdot}3997$$

Which is same as the predicted value.

$\therefore$ $y_6 = 1{\cdot}3997, y'_6 = 1{\cdot}6997,$ $[\because y' = x + y]$

Hence the table of results is as follows :

x	y	$y' = \frac{dy}{dx} = x + y$
$x_4 = {\cdot}20$	$y_4 = 1{\cdot}2428$	$y'_4 = 1{\cdot}4428$
$x_5 = {\cdot}25$	$y_5 = 1{\cdot}3180$	$y'_5 = 1{\cdot}5680$
$x_6 = {\cdot}30$	$y_6 = 1{\cdot}3997$	$y'_6 = 1{\cdot}6997$

Ans.

Objective Type Questions

1. Third approx. of differential eqn. $\frac{dy}{dx} = y^2 - x^2, y(0) = 1, h = 0{\cdot}1$ by Euler's method is : **(Indore 2013; Gwalior 13; Bhopal 13)**

(a) 1·2348 (b) 1·3348
(c) 1·3648 (d) 0·2348.

2. By modified Euler's method first approx. of differential equation $\frac{dy}{dx} = x + |\sqrt{y}|$, when $y(0) = 1, h = 0{\cdot}2$ is :

(a) 1·2205 (b) 1·2300
(c) 1·2290 (d) 1·2295.

3. By Picard method approx. at $y(0{\cdot}2)$ of differential equation $\frac{dy}{dx} = 1 + xy$, when $y(0) = 1, h = 0{\cdot}1$ is : **(Jabalpur 2016; Rewa 17)**

(a) 1·105 (b) 1·203
(c) 1·023 (d) None of these.

4. Picard method is useful when analytical solution is :

(a) Simple (b) Difficult
(c) None of these (d) Not any stage.

5. Taylor's method of solving differential equation is :

(a) Single step method (b) Multi step metod
(c) Neither (a) or (b) (d) None of these.

6. In differential equation $\frac{dy}{dx} = y^2 + 1, y(0) = 0$ if $h = 0{\cdot}2$ then by Taylor's method value of $y(0{\cdot}2)$ is :

(a) 0·02027 (b) 0·2027
(c) 1·2027 (d) 2·0271.

7. In differential equation $\frac{dy}{dx} = x + y, y(0) = 1, h = 0{\cdot}1$ value of $y = (0{\cdot}1)$ by Runge-kutta method is :

(a) 1·0103 (b) 1·003
(c) 1·1103 (d) None of these.

8. Value of k_4 in Runge-Kutta method is evaluated by :

(a) $hf\left(x_0+\frac{h}{2}, y_0+k_3\right)$ (b) $hf\left(x_0+\frac{h}{2}, y_0+\frac{k_3}{2}\right)$

(c) $hf(x_0+h, y_0+k_3)$ (d) $hf(x_0+3h, y_0+k_3)$.

9. Milne's predictor formula for the solution of $\frac{dy}{dx}=f(x,y)$ when $y(x_0)=y_0$ is :

(Indore 2011, 17; Jabalpur 15, 17; Bhopal 17)

(a) $y_4=y_0+\frac{h}{3}\left(2y_1'-y_2'+2y_3'\right)$ (b) $y_4=y_0+\frac{4h}{3}(2y_1'-y_2'+2y_3')$

(c) $y_4=y_0+\frac{h}{2}(2y_1'-y_2'+2y_3')$ (d) $y_4=y_2+\frac{4h}{3}(y_2'+4y_3'+y_4')$.

10. General form of Milne's corrector formula for the solution of $\frac{dy}{dx}=f(x,y)$, $y(x_0)=y_0$ is : **(Indore 2012, 14; Jabalpur 14, 17; Bhopal 14)**

(a) $y_{i-1}^c=y_{i-1}+\frac{h}{3}(y_{i-1}'+4y_i'+y_{i+1}')$ (b) $y_4=y_0+\frac{h}{3}(2y_1'-y_2'+2y_3')$

(c) $y_4=y_0+\frac{4h}{3}(2y_1'-y_2'+2y_3')$ (d) $y_4=y_0+\frac{h}{2}(2y_1'-y_2'+2y_3')$.

11. Milne's method for the solution of $\frac{dy}{dx}=f(x,y)$ least values of y prior to the desired value are :

(a) 4 (b) 3 (c) 2 (d) 1.

12. With initial condition $y=y_0$ at $x=x_0$ for $n=0,1,2,\ldots\ldots$ Euler's modified formula for the differential equation $\frac{dy}{dx}=f(x,y)$ is : **(Rewa 2011; Jabalpur 14)**

(a) $y_{n+1}=y_n+\frac{1}{2}hf(x_n,y_n)$ (b) $y_{n+1}=y_n-\frac{1}{2}hf(x_n,y_n)$

(c) $y_{n+1}=y_n-\frac{3}{2}hf(x_n,y_n)$ (d) $y_{n+1}=y_n+hf(x_n,y_n)$.

13. From Runge Kutta's formula of fourth order with usual representation of signs $y_{n+1}-y_n=\ldots\ldots\ldots$: **(Rewa 2011; Jabalpur 14)**

(a) $\frac{1}{6}(k_1+2k_2+3k_3+4k_4)$ (b) $\frac{1}{6}(k_1-2k_2+3k_3-4k_4)$

(c) $\frac{1}{6}(k_1-k_2+k_3-k_4)$ (d) $\frac{1}{6}(k_1+2k_2+2k_3+k_4)$.

14. Picard's method is : **(Bhopal 2011)**

(a) $y^{(n)}=y_0+\int_x^{x_0}f(x,y^{(n-1)})dx$ (b) $y^{(n)}=y_0+\int_{x_0}^{x}f(x,y^{(n-1)})dx$

(c) $y^{(n)}=y_0+\int_{x_0}^{x}f(x,y^{(n+1)})dx$ (d) $y^{(n)}=y_0+\int_x^{x_0}f(x,y^{(n+1)})dx$.

15. The method for solution of the differential equation $\frac{dy}{dx}=f(x,y), y(x)=y_0$: **(Ujjain 2011)**

(a) Picard's method (b) Runge's method

(c) Taylor series method (d) All of these.

16. An ordinary differential equation contains at least two independent variable. Is it true or false : **(Jabalpur 2011)**

17. In picard's method the formula $y^{(2)}=y_0+\int_{x_0}^{x}f(x,y^{(1)})dx$ used for : **(Jabalpur 2012)**

(a) For first approximation value (b) For second approximation

(c) For third approximation (d) None of these.

18. Milne's predictor formula for the solution of $\frac{dy}{dx}=f(x,y)$, when $y(x_0)=y_0$ is : **(Indore 2015, 16; Bhopal 16)**

(a) $y_u=y_0+\frac{h}{3}(2f_1-f_2+2f_3)$ (b) $y_u=y_0+\frac{4h}{3}(2f_1-f_2+2f_3)$

(c) $y_u=y_0+\frac{h}{2}(2f_1-f_2+2f_3)$ (d) None of these.

19. If $k_1=hF(x_0,y_0), k_2=hF(x_0+h,y_0+k_1)$, then Runge-kutta formula of second order is : **(Rewa 2015)**

(a) $y_1=k_1+\frac{1}{2}(k_1+k_2)$ (b) $y_1=y_0+\frac{1}{2}(k_1+k_2)$

(c) $y_1=y_0+\frac{1}{2}k_1$ (d) None of these.

20. By Runge-kutta's method of fourth order the value of y at the end of first sub-interval for the equation $\frac{dy}{dx}=f(x,y)$, with $y(x_0)=y_0$ is : **(Jabalpur 2015; Chhindwada 22)**

(a) $y_1=y_0+\frac{1}{6}(k_1+4k_2+k_3)$ (b) $y_1=y_0+\frac{1}{6}(k_1+2k_3+k_4)$

(c) $y_1=y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$ (d) None of these.

21. The value of k_2 in Runge-kutta's method : **(Bhopal 2015)**

(a) $hf(x, y)$ (b) $f(x, y)$ (c) $hf\left(x+\frac{1}{2}\right)$ (d) None of these.

22. Milne's method for the solution of $\frac{dy}{dx} = f(x, y)$ least value of y prior to the desirved value are : **(Rewa 2016)**

(a) 1 (b) 2 (c) 3 (d) 4.

23. By Euler's method n^{th} approximate value of y at $y_n, x = x_n = x_0 + nh$:

(Jabalpur 2017, 22; Indore 22)

(a) $y_n = y_0 + hf(x_{n-1}, y_{n-1})$ (b) $y_n = y_0 - hf(x_{n-1}, y_{n-1})$

(c) $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$ (d) None of these.

Ans. 1. (c), 2. (d), 3. (b), 4. (b), 5. (a), 6. (b), 7. (c), 8. (c), 9. (b), 10. (a), 11. (a), 12. (d), 13. (d), 14. (b), 15. (d), 16. False, 17. (b), 18. (b), 19. (b), 20. (c), 21. (d), 22. (d), 23. (c).

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