



Numerical Methods and Scientific Computation

[For B.Sc./B.A., IIIrd year
(Mathematics : Group-A, Paper-I)]

**A Unique Book
According to New Education Policy**



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Christian Eminent College,
Indore

New Edition



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NEW SYLLABUS

B.Sc./B.A. : Third year
NUMERICAL METHODS AND SCIENTIFIC COMPUTATION
(Mathematics : Group-A, Paper-I)
Course Code : S3-MATH1D

Unit-I

Methods of Solving Algebraic and Transcendental Equations : 1. Ramanujan, 2. Bisection, 3. Regula Falsi, 4. Secant, 5. Newton-Raphson.

Unit-II

Interpolation and Numerical Integration : 1. Lagrange interpolation, 2. Finite difference operators, 3. Interpolation formula using differences- (i) Gregory-Newton Forward Difference Interpolation, (ii) Gregory-Newton Backward Difference Interpolation. 4. Numerical Integration- (i) Newton- Cot's formulae, (ii) Trapezoidal rule, (iii) Simpson's 1/3 Rule, (iv) Simpson's 3/8 Rule, (v) Gauss Integration.

Unit-III

Methods to Solve of Linear Equations : 1. Direct methods for solving system of linear equations- (i) Gauss elimination, (ii) LU decomposition, (iii) Cholesky decomposition. 2. Iterative methods : (i) Jacobi, (ii) Gauss-Seidal.

Unit-IV

Numerical Solution of Ordinary Differential Equation : 1. Single step methods- (i) Picard, (ii) Taylor's series, (iii) Euler, (iv) Runge-Kutta. 2. Multistep methods- (i) Predictor-corrector, (ii) Modified Euler, (iii) Milne-Simpson.

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Unit-I

Methods of Solving Algebraic and Transcendental Equations : 1. Ramanujan, 2. Bisection, 3. Regula Falsi, 4. Secant, 5. Newton-Raphson.

Bisection Method or Bolzano Method

Let $f(x) = 0$ (1)

be given equation

Let $x = x_1$ and $x = x_2$ be two numbers such that $f(x_1)$ and $f(x_2)$ have opposite signs then there is at least one root between x_1 and x_2 say x_3 then

$$x_3 = \frac{x_1 + x_2}{2} \text{ be first approximate value of the root}$$

If $f(x_3) = 0$ then x_3 be root of equation (1) otherwise

Case I : If $f(x_1) f(x_3) < 0$ then

$$x_4 = \frac{x_1 + x_3}{2} \text{ be second approximate value of the root}$$

Case II : If $f(x_2) f(x_3) < 0$ then

$$x_4 = \frac{x_2 + x_3}{2} \text{ be second approximate value of the root}$$

If $f(x_4) = 0$ then x_4 be root of the equation (1) otherwise continue above process until the root is obtained to the desired accuracy.

Q.1. Find a real root of the equation $x^2 - 5x + 2 = 0$ by bisection method.

Sol : Let $f(x) = x^2 - 5x + 2$ (*)

$$\text{Let } x_1 = 0 \Rightarrow f(0) = 0 - 0 + 2 \Rightarrow f(0) = 2 > 0$$

$$\text{Let } x_2 = 1 \Rightarrow f(1) = 1 - 5 + 2 \Rightarrow f(1) = -2 < 0$$

$$\therefore f(x_1) f(x_2) = f(0) f(1) < 0$$

then by bisection method,

$$x_3 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

$$\text{By (*) } f(x_3) = f(0.5) = (0.5)^2 - 5(0.5) + 2 = -0.25 < 0$$

$$\therefore f(0) f(0.5) < 0$$

$$\therefore x_4 = \frac{0+0.5}{2} = 0.25$$

[1]

By (*)

$$\therefore f(x_4) = f(0.25) = (0.25)^2 - 5(0.25) + 2 = 0.8125 > 0$$

$$\Rightarrow f(x_4) f(x_3) = f(0.25) f(0.5) < 0$$

$$\therefore x_5 = \frac{0.25 + 0.5}{2} = 0.375$$

By *

$$\therefore f(x_5) = f(0.375) = (0.375)^2 - 5(0.375) + 2 = 0.265625 > 0$$

$$\therefore f(x_5) f(x_3) = f(0.375) f(0.5) < 0$$

$$x_6 = \frac{0.375 + 0.5}{2} = 0.4375$$

By *

$$f(x_6) = f(0.4375) = (0.4375)^2 - 5(0.4375) + 2 = 0.003906 > 0$$

$$\therefore f(x_6) f(x_3) < 0$$

$$\therefore x_7 = \frac{0.4375 + 0.5}{2} = 0.46875$$

$\therefore$ root $x = 0.46875$ approximate

Q.2. Find a root of the equations $x^3 - x - 4 = 0$

Sol: Let $f(x) = x^3 - x - 4$

$$\text{Let } x_1 = 1 \Rightarrow f(1) = -4 < 0$$

$$x_2 = 2 \Rightarrow f(2) = 2 > 0$$

$\therefore$ root x_3 may be between $x_1 = 1$ and $x_2 = 2$ then by bisection method

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 2}{2} = 1.5$$

be first approximate root

$$\text{By } * \quad f(x_3) = f(1.5) = (1.5)^3 - 1.5 - 4 = -2.125 < 0$$

$$\therefore f(x_3) f(x_2) < 0 \text{ then}$$

$$x_4 = \frac{x_3 + x_2}{2} = \frac{1.5 + 2}{2} = 1.75$$

$$\text{By } * \quad f(x_4) = f(1.75) = -0.390625 < 0$$

$$\therefore f(x_4) f(x_2) < 0 \text{ then}$$

$$x_5 = \frac{x_4 + x_2}{2} = \frac{1.75 + 2}{2} = 1.875$$

$$\Rightarrow f(x_5) = f(1.875) = 0.71696 > 0$$

$$\therefore f(x_4) f(x_5) < 0 \text{ then}$$

$$x_6 = \frac{1.75 + 1.875}{2} = 1.8125$$

$$\text{By } * \quad f(x_6) = f(1.8125) = 0.141845 > 0$$

$$\therefore f(x_4) f(x_6) < 0 \text{ then}$$

$$x_7 = \frac{x_4 + x_6}{2} = \frac{0.175 + 1.8125}{2} = 1.78125$$

$$\text{By } * \quad f(x_7) = f(1.78125) = -0.1296 < 0$$

$$\therefore f(x_7) f(x_6) < 0$$

$$\text{then } x_8 = \frac{x_7 + x_6}{2} = \frac{1.78125 + 1.8125}{2} = 1.796875$$

$$\therefore x_7 \approx x_8$$

$\therefore x = 1.796875$ be approximate root of *

Q.13. Find a real root of the following equation by bisection method.

$$(1) x^3 - 2x - 5 = 0$$

$$(2) x^3 - x - 1 = 0$$

$$(3) x^3 - x - 4 = 0$$

Sol: (1) Given that

$$f(x) = x^3 - 2x - 5 \quad \dots (*)$$

$$\text{Let } x_1 = 2 \Rightarrow f(2) = -1 < 0 \text{ By } *$$

$$\text{Let } x_2 = 3 \Rightarrow f(3) = 16 > 0 \text{ By } *$$

$$\therefore f(x_1) f(x_2) < 0$$

$\Rightarrow$ root of eq. (*) may be between $x_1 = 2$ and $x_2 = 3$ then by bisection method

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2 + 3}{2} = 2.5 \text{ be first approximate root}$$

$$\text{By } * \quad f(x_3) = f(2.5) = 5.625 > 0,$$

$$\therefore f(x_1) f(x_3) < 0 \text{ then}$$

$$x_4 = \frac{x_1 + x_3}{2} = \frac{2 + 2.5}{2} = 2.25 \text{ be second approximate root}$$

$$\text{By } * \quad f(x_4) = 1.89 > 0$$

$$\therefore f(x_1) f(x_4) < 0 \text{ then}$$

$$x_5 = \frac{x_1 + x_4}{2} = \frac{2 + 2.25}{2} = 2.125 \text{ be third approximate root}$$

$$\text{By } * \quad f(x_5) = 0.346 > 0$$

$$\therefore f(x_1) f(x_5) < 0 \text{ then}$$

$$x_6 = \frac{x_1 + x_5}{2} = \frac{2 + 2.125}{2} = 2.0625$$

$$\text{By } * \quad f(x_6) = f(2.0625) = -0.35 < 0$$

$$\therefore f(x_6) f(x_5) < 0 \text{ then}$$

$$x_7 = \frac{x_6 + x_5}{2} = \frac{2.0625 + 2.125}{2} = 2.09375 \text{ be } 5^{\text{th}} \text{ approximate value}$$

$$\text{By } * \quad f(x_7) = -0.008 < 0$$

$$\therefore f(x_7) f(x_5) < 0 \text{ then}$$

$$x_8 = \frac{x_7 + x_5}{2} = \frac{2.09375 + 2.125}{2} = 2.109375$$

be sixth approximate value

$$\therefore f(x_8) = f(2.109375) = 0.166836 > 0, \text{ then}$$

$$\therefore f(x_7) f(x_8) < 0$$

$$x_9 = \frac{x_7 + x_8}{2} = \frac{2.09375 + 2.19375}{2} = 2.1015$$

$$\therefore x_8 \equiv x_9$$

$$\therefore x = 2.1015 \text{ be approximate root of } f(x) = 0$$

(2) Given that

$$f(x) = x^3 - x - 1 \quad \dots (*)$$

$$\text{By } (*) \quad f(x_1) = f(1) = -1 < 0$$

$$f(x_2) = f(2) = 5 > 0$$

$\therefore f(x_1) f(x_2) < 0$ then root may be between $x_1 = 1$ and $x_2 = 2$, then by bisection method

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 2}{2} = 1.5 \text{ be first approximate root}$$

$$\text{By } (*) \quad f(x_3) = f(1.5) = 0.875 > 0$$

$$\therefore f(x_1) f(x_3) < 0 \text{ then}$$

$$x_4 = \frac{x_1 + x_3}{2} = \frac{1 + 1.5}{2} = 1.25 \text{ be second approximate root}$$

$$\therefore \text{By } (*) \quad f(x_4) = f(1.25) = -0.296875 < 0$$

$$\therefore f(x_4) f(x_3) < 0 \text{ then}$$

$$x_5 = \frac{x_4 + x_3}{2} = \frac{1.25 + 1.5}{2} = 1.375 \text{ be third approximate root}$$

$$\text{By } (*) \quad f(x_5) = f(1.375) = +0.2246 > 0$$

$$\therefore f(x_5) f(x_4) < 0 \text{ then}$$

$$x_6 = \frac{x_5 + x_4}{2} = \frac{1.375 + 1.25}{2} = 1.3125 \text{ be fourth approximate root}$$

$$\text{By } (*) \quad f(x_6) = f(1.3125) = -0.05151 < 0$$

$$\therefore f(x_6) f(x_5) < 0 \text{ then}$$

$$x_7 = \frac{x_6 + x_5}{2} = \frac{1.3125 + 1.375}{2} = 1.34375 \text{ be } 5^{\text{th}} \text{ approximation}$$

$$\text{By } (*) \quad f(x_7) = f(1.34375) = 0.08261 > 0$$

$$\therefore f(x_6) f(x_7) < 0 \text{ then}$$

$$x_8 = \frac{x_6 + x_7}{2} = \frac{1.3125 + 1.34375}{2} = 1.32815$$

be 6th approximate root

$$\text{By } (*) \quad f(x_8) = f(1.32815) = 0.0145 > 0$$

$$\therefore f(x_6) f(x_8) < 0 \text{ then}$$

$$x_9 = \frac{x_6 + x_8}{2} = \frac{1.3125 + 1.38815}{2} = 1.320313$$

$$\therefore x_8 \equiv x_9 \text{ then}$$

$$x = 1.320313 \text{ be approximate root of } f(x) = 0$$

(3) Given that

$$f(x) = x^3 - x - 4 \quad \dots (*)$$

$$\text{Let } x_1 = 1 \text{ then by } (*), f(x_1) = f(1) = -3 < 0$$

$$\& \quad x_2 = 2 \text{ then by } (*), f(x_2) = f(2) = 2 > 0$$

$\therefore f(x_1) f(x_2) < 0$ then root may be between $x_1 = 1$ and $x_2 = 2$ of equation *, then by bisection method

$$x_3 = \frac{x_1 + x_2}{2} = \frac{1 + 2}{2} = 1.5$$

$$\text{By } (*) \quad f(1.5) = -2.125 < 0$$

$$\therefore f(x_3) f(x_2) < 0 \text{ then}$$

$$x_4 = \frac{x_3 + x_2}{2} = \frac{1.5 + 2}{2} = 1.75$$

be second approximate root

$$\text{By } (*) \quad f(x_4) = f(1.75) = -0.3906 < 0$$

$$\therefore f(x_4) f(x_2) < 0 \text{ then}$$

$$x_5 = \frac{x_4 + x_2}{2} = \frac{1.75 + 2}{2} = 1.875$$

$$\text{By } (*) \quad f(x_5) = f(1.875) = 0.7167 > 0$$

$$\therefore f(x_4) f(x_5) < 0 \text{ then}$$

$$x_6 = \frac{x_4 + x_5}{2} = \frac{1.75 + 1.875}{2} = 1.8125$$

$$\text{By } (*) \quad f(x_6) = f(1.8125) = 0.1418 > 0$$

$$\therefore f(x_4) f(x_6) < 0 \text{ then}$$

$$x_7 = \frac{x_4 + x_6}{2} = \frac{1.75 + 1.8125}{2} = 1.78125$$

$$\text{By } (*) \quad f(x_7) = f(1.78125) = -0.12968 < 0$$

$$\therefore f(x_7) f(x_6) < 0 \text{ then}$$

$$x_8 = \frac{x_7 + x_6}{2} = \frac{1.78125 + 1.8125}{2} = 1.796875$$

By (*) $f(x_8) = f(1.796875) = 0.0048 < 0$
 $\therefore f(x_7) f(x_8) < 0$ then

$$x_9 = \frac{x_7 + x_8}{2} = \frac{1.78125 + 1.796875}{2} = 1.790$$

$\therefore x_8 \equiv x_9$
 $\Rightarrow x = 1.790$ be approximate root of $f(x) = 0$

Regula Falsi Method OR Method of False position

Let $f(x) = 0$ be given equation

Let $x = x_1$ and $x = x_2$ be two numbers such that $f(x_1)$ and $f(x_2)$ have opposite signs then there is atleast one root between x_1 and x_2 say x_3 then

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad (*) \text{ be first approximate root of } f(x) = 0$$

find the value of $f(x_3)$; If $f(x_3) = 0$ then x_3 be root of the equation (1) otherwise

Case I : If $f(x_1) f(x_3) < 0$ then by (*)

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} \quad \text{be second approximate root of } f(x) = 0$$

Case II : If $f(x_2) f(x_3) < 0$ then by (*)

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} \quad \text{be second approximate root of } f(x) = 0$$

If $f(x_4) = 0$ then x_4 be root of equation (1), otherwise continue above process until the root is obtained to the desired accuracy. ■

Q.4 Find a real root of the following equation by Regula Falsi Method.

$$(1) x^3 - 2x - 5 = 0 \quad (2) x^3 - 5x - 7 = 0$$

$$(3) x^3 - 5x + 3 = 0 \quad (4) x^3 - 9x + 1 = 0$$

Sol : (1) Let $f(x) = x^3 - 2x - 5$... (*)

Let $x_1 = 2 \Rightarrow f(2) = -1 < 0$ By (*)

& for $x_2 = 3 \Rightarrow f(3) = 16 > 0$ By (*)

$\therefore f(x_1) f(x_2) < 0$ then root of equation (1) may be between $x_1 = 2$ and $x_2 = 3$, then by Regula Falsi Method.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.0588$$

be first approximate root

By (*) $f(x_3) = f(2.0588) = -0.39105 < 0$

$\therefore f(x_2) f(x_3) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = \frac{(2.0588)(16) - 3(-0.39105)}{16 - (-0.39105)}$$

$x_4 = 2.08125$ be second approximate root

By (*) $f(x_4) = f(2.08125) = -0.14175 < 0$

$\therefore f(x_2) f(x_4) < 0$ then $x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)}$

$$x_5 = \frac{2.08125(16) - 3(-0.14175)}{16 - (-0.14175)} = 2.089318$$

be third approximate root

By (*) $f(x_5) = f(2.089318) = -0.0582413 < 0$

$\therefore f(x_2) f(x_5) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = \frac{2.089318(16) - 3(-0.0582413)}{16 - (-0.0582413)}$$

$x_6 = 2.09262$ be fourth approximate root

By (*) $f(x_6) = f(2.09262) = -0.021523 < 0$

$\therefore f(x_2) f(x_6) < 0$ then

$$x_7 = \frac{x_6 f(x_2) - x_2 f(x_6)}{f(x_2) - f(x_6)} = 2.09383$$

$\therefore x_6 \equiv x_7$

then $x = 2.09$ be approximate root of $f(x)$

(2) Given that $f(x) = x^3 - 5x - 7$... (*)

Let $x_1 = 2$ then by $f(x_1) = f(2) = -9 < 0$

& for $x_2 = 3$ then by (*) by $f(x_2) = f(3) = 5 > 0$

$\therefore f(x_1) f(x_2) < 0$ then root of $f(x)$ may be between $x_1 = 2$ and $x_2 = 3$ then by regula falsi method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(5) - 3(-9)}{5 - (-9)} = \frac{37}{14} = 2.643$$

be first approximate root

By (*) $f(x_3) = f(2.643) = -1.75246 < 0$... (1)

$\therefore f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 2.736$$

By (*) $f(x_4) = f(2.736) = -0.199 < 0$

$\therefore f(x_4) f(x_2) < 0$ then

$$x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)} = 2.746$$

be third approximate root

By (*) $f(x_5) = f(2.746) = -0.024 < 0$

$\therefore f(x_5) f(x_2) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = 2.747$$

be fourth approximate root

By (*) $f(x_6) = f(2.747) = -0.006 < 0$

$\therefore f(x_6) f(x_2) < 0$ then

$$x_7 = \frac{x_6 f(x_2) - x_2 f(x_6)}{f(x_2) - f(x_6)} = 2.747$$

$\therefore x_6 = x_7$ then $x = 2.747$ be root of equation (*)

(3) Given that

$$f(x) = x^3 - 5x + 3$$

Let $x_1 = 1$ then by (*) $f(x_1) = f(1) = -1$

& for $x_2 = 2$ then by (*) $f(x_2) = f(2) = 1$

$\therefore f(x_1) f(x_2) < 0$ then root of (*) may be between $x_1 = 1$ and $x_2 = 2$ then by false position (regula-falsi method)

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1(1) - 2(-1)}{1 - (-1)}$$

$\Rightarrow x_3 = 1.5$ be first approximate root

By (*) $f(x_3) = f(1.5) = -1.125 < 0$

$\therefore f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 1.7647$$

be second approximate root

By (*), $f(x_4) = f(1.7647) = -0.3279 < 0$

$\therefore f(x_4) f(x_2) < 0$ then

$$x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)} = 1.823$$

By (*) $f(x_5) = f(1.823) = -0.0576 < 0$

$\therefore f(x_5) f(x_2) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = 1.8324$$

be fourth approximate root

By (*) $f(x_6) = f(1.8324) = -0.009694 < 0$

$\therefore f(x_6) f(x_2) < 0$ then

$$x_7 = \frac{x_6 f(x_2) - x_2 f(x_6)}{f(x_2) - f(x_6)} = 1.8324$$

$\therefore x_6 = x_7$

$\Rightarrow x = 1.8324$ be an approximate root of $f(x) = 0$

(4) Given that $f(x) = x^3 - 9x + 1$

...(*)

Let $x_1 = 2$ then by (*) $f(x_1) = f(2) = -9$

& for $x_2 = 3$ then by (*) $f(x_2) = f(3) = 1$

$\therefore f(x_1) f(x_2) < 0$

then root may be between $x_1 = 2$ and $x_2 = 3$ then by regula falsi method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{2(1) - 3(-9)}{1 - (-9)} = \frac{29}{10}$$

$\Rightarrow x_3 = 2.9$ be first approximate root

By (*) $f(x_3) = f(2.9) = -0.711 < 0$

$\therefore f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 2.9416$$

be second approximate root

By (*), $f(x_4) = f(2.9416) = -0.0207 < 0$

$\therefore f(x_4) f(x_2) < 0$ then

$$x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)} = 2.9428$$

be third approximate root

By (*), $f(x_5) = f(2.9428) = -0.0003 < 0$

$\therefore f(x_5) f(x_2) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = 2.9428$$

$\therefore x_5 = x_6$

$\therefore x = 2.948$ be an approximate root of $f(x) = 0$

Q.5. Find the smallest positive root of the equation $x^4 - x - 10 = 0$ by using regula falsi method.

Sol: Let $f(x) = x^4 - x - 10$

...(*)

Let $x > 0$ be a positive root of equation (*)

Let $x_1 = 1 > 0$ then by (*) $f(x_1) = f(1) = -10$

for $x_2 = 2 > 0$ then by (*) $f(x_2) = f(2) = 4$

$\because f(x_1) f(x_2) < 0$ then root of equation (*) may be between $x_1 = 1$ and $x_2 = 2$, then by regula falsi method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{1 \cdot (4) - 2 \cdot (-10)}{4 - (-10)} = \frac{24}{14} = 1.7143$$

be first approximate root

By (*) $f(x_3) = f(1.7143) = -3.0776 < 0$

$\because f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 1.8385$$

be second approximate root

By (*) $f(x_4) = f(1.8385) = -0.4135 < 0$

$\because f(x_4) f(x_2) < 0$ then

$$x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)} = 1.8512$$

be third approximate

By (*) $f(x_5) = f(1.8512) = -0.10727 < 0$

$\because f(x_5) f(x_2) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = 1.855$$

be fourth approximate root

$\therefore x_5 = x_6$

$\therefore x = 1.855$ be an approximate root of $f(x) = 0$

Q.6. Apply regula falsi method to solve the equation

$$3x - \cos x - 1 = 0$$

Sol : Let $f(x) = 3x - \cos x - 1$...(*)

Let $x_1 = 0.60$ then by (*), $f(x_1) = f(0.6) = -0.02534 < 0$ and for $x_2 = 0.61$ then by (*) $f(x_2) = f(0.61) = 0.01035 > 0$

$\because f(x_1) f(x_2) < 0$ then root of equation $f(x) = 0$ may be between $x_1 = 0.60$ and $x_2 = 0.61$ then by Regula falsi method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.6071$$

By (*) $f(x_3) = f(0.6071) = -0.00001 < 0$

$\because f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 0.6071$$

$\therefore x_3 = x_4$

$\therefore x = 0.6071$ be an appropriate root of $f(x) = 0$

Q.7. Determine the root of $xe^x - 2 = 0$ by method of false position

Sol : Let $f(x) = xe^x - 2$...(*)

Let $x_1 = 0$ then by (*) $f(x_1) = f(0) = -2 < 0$

and $x_2 = 1$ then by (*) $f(x_2) = f(1) = 0.718282 > 0$

$\because f(x_1) f(x_2) < 0$ then root may be between $x_1 = 0$ and $x_2 = 1$,

then by Regula falsi (false position method)

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.735759$$

be first approximate root

By (*), $f(x_3) = f(0.735759) = -0.464423 < 0$

$\because f(x_3) f(x_2) < 0$ then

$$x_4 = \frac{x_3 f(x_2) - x_2 f(x_3)}{f(x_2) - f(x_3)} = 0.839521$$

be second approximate root

By (*) $f(x_4) = f(0.839521) = -0.056293 < 0$

$\therefore f(x_4) f(x_2) < 0$ then

$$x_5 = \frac{x_4 f(x_2) - x_2 f(x_4)}{f(x_2) - f(x_4)} = 0.851184$$

be third approximate root

By (*) $f(x_5) = f(0.851184) = -0.006171 < 0$

$\therefore f(x_5) f(x_2) < 0$ then

$$x_6 = \frac{x_5 f(x_2) - x_2 f(x_5)}{f(x_2) - f(x_5)} = 0.852452$$

$\therefore x_5 = x_6$

$\therefore x = 0.852$ be an approximate root of equation $f(x) = 0$

Secant Method

Let $f(x) = 0$...(*)

be given equation. Let $x = x_1$ and $x = x_2$

be any two values and find $f(x_1)$ and $f(x_2)$, it is not necessary that $f(x_1) f(x_2) < 0$, then by secant method.

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \text{ be first approximate root and}$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

be second approximate root and so.

$$x_{n+1} = \frac{x_n f(x_n) - x_{n-1} f(x_{n-1})}{f(x_n) - f(x_{n-1})} \text{ be an approximate root}$$

Q.8. Find a root of $x^2 - 4x - 10 = 0$ by the secant method.

Sol : Let $f(x) = x^2 - 4x - 10$...(*) be given equation

Let $x_1 = 4$ then by (*) $f(x_1) = f(4) = -10$

& for $x_2 = 2$ then by (*) $f(x_2) = f(2) = -14$

then by secant method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 9 \text{ be first approximate root}$$

By (*), $f(x) = f(9) = 35$

$\therefore$ By secant method

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 4$$

be second approximate root

By (*) $f(x_4) = f(4) = -10$ then by secant method

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = 5.1111 \text{ be an approximate root}$$

By (*) $f(x_5) = f(5.1111) = -4.3211$ then by secant method

$$x_6 = \frac{x_4 f(x_5) - x_5 f(x_4)}{f(x_5) - f(x_4)} = 5.9565$$

be an approximate root

By (*) $f(x_6) = f(5.9565) = 1.6539$ then by secant method

$$x_7 = \frac{x_5 f(x_6) - x_6 f(x_5)}{f(x_6) - f(x_5)} = 5.7275$$

be an approximate root

By (*) $f(x_7) = f(5.7274) = -0.143$ then by secant method

$$x_8 = \frac{x_6 f(x_7) - x_7 f(x_6)}{f(x_7) - f(x_6)} = 5.7411$$

be an approximate root

By (*) $f(x_8) = f(5.7411) = -0.0042$ then by secant method

$$x_9 = \frac{x_7 f(x_8) - x_8 f(x_7)}{f(x_8) - f(x_7)} = 5.7416$$

be an approximate root

$\therefore x_8 = x_9$ then

$x = 5.741$ be an approximate root

Q. 9. Find a root of equation $x^3 - 2x - 5 = 0$ by secant method :

Sol. Let $f(x) = x^3 - 2x - 5$ be given equation ...(*)

Let $x_1 = 2$ then by $f(x_1) = f(2) = -1$ & $x_2 = 3$ then by (*) $f(x_2) = f(3) = 16$ then by secant method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.058824$$

be an approximate root

By (*), $f(x_3) = f(2.058824) = -0.390795$ then by secant method

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 2.081264$$

By (*) $f(x_4) = f(2.081264) = -0.1472$

then by secant method

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = 2.045264$$

$\therefore x_4 = x_5$ then

$$x = 2.0452$$

be an approximate root

Q.10. Find a root of equation $x - e^{-x} = 0$, correct to three decimal places by the secant method.

Sol : Let $f(x) = x - e^{-x}$...(*)

be given equation

Let $x_1 = 0$ then by (*) $f(x_1) = f(0) = -1$ and for $x_2 = 1$ then by (*) $f(x_2) = f(1) = 0.6321$

then by secant method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.6127$$

be an approximate root

By (*) $f(x_3) = f(0.6127) = 0.0708$

then by secant method

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 0.5638 \text{ be second}$$

approximate

By (*) $f(x_4) = f(0.5638) = -0.0052$ then by secant method

$$x_5 = \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = 0.5671$$

By (*) $f(x_2) = f(0.5671) = -0.0001$

∴ By secant method

$$x_4 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_2) - f(x_1)} = 0.5672$$

∴ $x_5 = x_4$ then $x = 0.567$ be an approximate root of $f(x) = 0$ **Newton-Raphson Method**Let $f(x) = 0$ (*) be given equationLet $x = x_0$ be any point then by Newton-Raphson Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \text{ be first approximate root and}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

be second approximate root and in general

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \text{ be } n^{\text{th}} \text{ approximate root}$$

Q.11 Find a real root of the following equation by Newton-Raphson Method :

(1) $x^3 - 5x + 2 = 0$ (2) $x^3 - 3x + 1 = 0$

(3) $2x^3 - 3x - 6 = 0$ (4) $x^3 - 3x - 5 = 0$

Sol : (1) Let $f(x) = x^3 - 5x + 2$

$$f'(x) = 3x^2 - 5$$

Let $x = x_0 = 0$ then by (1) & (2)

$$f(0) = 2$$

$$f'(0) = -5$$

then by Newton Raphson Method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(2)}{(-5)} = \frac{2}{5} = 0.4$$

be first approximate root

Now, By Newton-Raphson's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5x_n + 2}{2x_n^2 - 5}$$

$$x_{n+1} = \frac{x_n^3 - 2}{2x_n^2 - 5}$$

$$\text{Put } x_0 = 0.4 \text{ then by (*) } x_1 = \frac{x_0^3 - 2}{2x_0^2 - 5}$$

$$x_1 = \frac{(0.4)^3 - 2}{2(0.4)^2 - 5} = 0.438095$$

$$\text{By (*) } x_2 = \frac{x_1^3 - 2}{2x_1^2 - 5} = \frac{(0.438095)^3 - 2}{2(0.438095)^2 - 5} = 0.4380$$

$$\text{By (*) } x_3 = \frac{x_2^3 - 2}{2x_2^2 - 5} = \frac{(0.438447)^3 - 2}{2(0.438447)^2 - 5} = 0.4380$$

$$x_2 = x_3$$

 $x = 0.438447$ be an approximate root

(2) Let $f(x) = x^3 - 3x + 1$... (1)

$$\Rightarrow f(x) = 3x^2 - 3$$
 ... (2)

By (1) $f(0) = 1$ and $f(1) = -1$

Let $x_0 = \frac{0+1}{2} = 0.5$

and by Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 3}$$

$$\text{By (*) } x_1 = \frac{2x_0^3 - 1}{3x_0^2 - 3} = \frac{-0.75}{-2.25} = 0.3333$$

$$\text{By (*) } x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3}$$

$$\Rightarrow x_2 = \frac{2(0.3333)^3 - 1}{3(0.3333)^2 - 3} = 0.3472$$

$$\text{By (*) } x_3 = \frac{2x_2^3 - 1}{3x_2^2 - 3} = \frac{2(0.3472)^3 - 1}{3(0.3472)^2 - 3}$$

$$x_3 = 0.3473$$

$$\text{By (*) } x_4 = \frac{2x_3^3 - 1}{3x_3^2 - 3} = \frac{2(0.3473)^3 - 1}{3(0.3473)^2 - 3} = 0.3473$$

$x_3 \equiv x_4$
 $x = 0.3473$ be an approximate root.

(3) Let $f(x) = 2x^3 - 3x - 6$

& $f'(x) = 6x^2 - 3$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n^3 - 3x_n - 6}{6x_n^2 - 3}$$

$$x_{n+1} = \frac{4x_n^3 + 6}{6x_n^2 - 3}$$

Let $x_0 = 2$ then by (*)

$$x_1 = \frac{4x_0^3 + 6}{6x_0^2 - 3} = \frac{4(2)^3 + 6}{6(2)^2 - 3} = \frac{38}{21} = 1.809524$$

$$\text{By (*) } x_2 = \frac{4x_1^3 + 6}{6x_1^2 - 3} = \frac{4(1.809524)^3 + 6}{6(1.809524)^2 - 3} = 1.7842$$

$$\text{By (*) } x_3 = \frac{4x_2^3 + 6}{6x_2^2 - 3} = \frac{4(1.7842)^3 + 6}{6(1.7842)^2 - 3} = 1.783769$$

$$\text{By (*) } x_4 = \frac{4x_3^3 + 6}{6x_3^2 - 3} = \frac{4(1.783769)^3 + 6}{6(1.783769)^2 - 3} = 1.783769$$

$x_3 \equiv x_4$
 then $x = 1.783769$ be an approximate root

(4) Let $f(x) = x^3 - 3x - 5$

$\Rightarrow f(x) = 3x^2 - 3$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3}$$

$$x_{n+1} = \frac{2x_n^3 + 5}{3x_n^2 - 3}$$

2 then by (*)

$$x_1 = \frac{2x_0^3 + 5}{3x_0^2 - 3} = \frac{21}{9} = 2.3333$$

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 3} = \frac{2(2.3333)^3 + 5}{3(2.3333)^2 - 3} = \frac{30.40632}{13.3329}$$

$$x_2 = 2.2805$$

$$\text{By (*) } x_3 = \frac{2x_2^3 + 5}{3x_2^2 - 3} = \frac{2(2.2805)^3 + 5}{3(2.2805)^2 - 3} = \frac{28.86099}{12.66366}$$

$$x_3 = 2.2790$$

$$\text{By (*) } x_4 = \frac{2x_3^3 + 5}{3x_3^2 - 3} = \frac{2(2.279)^3 + 5}{3(2.279)^2 - 3} = \frac{28.673526}{12.58152}$$

$$x_4 = 2.2790$$

$$x_3 = x_4$$

$x = 2.279$ be an approximate root

Q.12 Compute the positive root of the equation $x^4 - x - 10 = 0$ by Newton-Raphson Method

Sol. Let $f(x) = x^4 - x - 10$

$\Rightarrow f(x) = 4x^3 - 1$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1}$$

$$x_n = \frac{3x_n^4 + 10}{4x_n^3 - 1}$$

Let $x_0 = 2$ then by (*)

$$x_1 = \frac{3(2)^4 + 10}{4(2)^3 - 1} = \frac{58}{31} = 1.87097$$

$$\text{By (*) } x_2 = \frac{3x_1^4 + 10}{4x_1^3 - 1} = \frac{3(1.87097)^4 + 10}{4(1.87097)^3 - 1} = 1.85579$$

$$\text{By (*) } x_3 = \frac{3x_2^4 + 10}{4x_2^3 - 1} = \frac{3(1.85579)^4 + 10}{4(1.85579)^3 - 1} = 1.85558$$

$$x_2 = x_3$$

$x = 1.8555$ be an approximate root

Q.13. Find the real positive root of $3x - \cos x - 1 = 0$ by Newton-Raphson Method :

Sol. : Let $f(x) = 3x - \cos x - 1$

$\Rightarrow f'(x) = 3 + \sin x$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{\sin x_n + 3}$$

$$x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{\sin x_n + 3} \quad \dots(*)$$

Let $x_0 = 0.5$
then by (*)

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{\sin x_0 + 3} = \frac{(0.5) \sin 0.5 + \cos 0.5 + 1}{\sin 0.5 + 3} = \frac{2.11729533}{3.479425539}$$

$$x_1 = 0.608518649$$

$$\text{By } (*) \quad x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{\sin x_1 + 3} = \frac{2.16835703}{3.57165265} = 0.607101878$$

$$\text{By } (*) \quad x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{\sin x_2 + 3} = \frac{2.16765013}{3.57048962} = 0.607101878$$

$$\therefore x_2 = x_3$$

then $x = 0.607102$ be an approximate root

Q.14. Find a real root of $4x - e^x = 0$ that lies between 2 and 3 by Newton's Method.

Sol. : Let $f(x) = 4x - e^x$...(1)

$$\Rightarrow f'(x) = 4 - e^x \quad \dots(2)$$

By Newton-Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{4x_n - e^{x_n}}{4 - e^{x_n}}$$

$$x_{n+1} = \frac{(1 - x_n)e^{x_n}}{4 - e^{x_n}} \quad \dots(*)$$

$$\text{Given that } x = 2 \Rightarrow f(2) = 0.6109$$

$$\& \quad x = 3 \Rightarrow f(3) = -8.0855$$

$$\therefore f(2) f(3) < 0 \text{ then root may be between 2 and 3}$$

Let $x_0 = 2$ be an approximate root

$$\text{By } (*) \quad x_1 = \frac{(1 - x_0)e^{x_0}}{4 - e^{x_0}} = 2.18026963$$

$$\text{By } (*) \quad x_2 = \frac{(1 - x_1)e^{x_1}}{4 - e^{x_1}} = 2.14444312$$

$$\text{By } (*) \quad x_3 = \frac{(1 - x_2)e^{x_2}}{4 - e^{x_2}} = 2.15336627$$

$$\& \text{ By } (*) \quad x_4 = \frac{(1 - x_3)e^{x_3}}{4 - e^{x_3}} = 2.15329237$$

$$\therefore x_3 = x_4$$

$$x = 2.15329 = 2.1533 \text{ be an approximate root}$$

Q.15. Find the cube root of 2 approximate by Newton's Method :

$$\text{Sol. : Let } x = (2)^{1/3}$$

$$\text{or } x^3 = 2 \Rightarrow x^3 - 2 = 0$$

$$\text{Let } f(x) = x^3 - 2 \quad \dots(1)$$

$$\Rightarrow f'(x) = 3x^2 \quad \dots(2)$$

By Newton's Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2} \quad \dots(*)$$

$$\text{By (1) } f(1) = -1$$

$$\& \quad f(2) = 6$$

$$\therefore f(1) f(2) < 0, \text{ then root may be between 1 and 2}$$

Let $x_0 = 1.5$ then by (*)

$$x_1 = \frac{2x_0^3 + 2}{3x_0^2} = 1.2963$$

$$\text{By } (*) \quad x_2 = \frac{2x_1^3 + 2}{3x_1^2} = 1.26093$$

$$\text{By } (*) \quad x_3 = \frac{2x_2^3 + 2}{3x_2^2} = 1.25992$$

$$\text{By } (*) \quad x_4 = \frac{2x_3^3 + 2}{3x_3^2} = 1.25992$$

$$\therefore x_3 = x_4 \text{ then}$$

$$x = 1.2599 \text{ be an approximate root}$$

Objectives

- If the function $f(x)$ is continuous on $[a, b]$ and $f(a)f(b) < 0$ then the equation $f(x) = 0$ has roots in the interval (a, b)
 - An odd number of roots
 - An even number of roots
 - At least one real root or an odd number of roots
 - None of these

2. The first approximation to the root of $f(x) = 0$ in bisection method is :

(a) $\frac{x_1 + x_2}{2}$ if $f(x_1) f(x_2) < 0$ (b) $\frac{x_1 + x_2}{2}$ if $f(x_1) f(x_2) > 0$
 (c) $\frac{x_1 + x_2}{4}$ if $f(x_1) f(x_2) < 0$ (d) $\frac{x_1 - x_2}{2}$ if $f(x_1) f(x_2) < 0$

3. To find out the root of the equation $f(x) = 0$ by secant method; the formula used is :

(a) $x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$
 (b) $x_{n+1} = x_n + \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n)$
 (c) $x_{n+1} = x_n - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} f(x_n)$
 (d) $x_{n+1} = x_n + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} f(x_n)$

4. The formula used for obtaining root of the equation $f(x) = 0$ by Newton Raphson Method is :

(a) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (b) $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$
 (c) $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$ (d) $x_{n+1} = x_n + \frac{f'(x_n)}{f(x_n)}$

5. To find out the root of the equation $f(x) = 0$ by Regula Falsi Method, the formula used is :

(a) $x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n)$

(b) $x_{n+1} = x_n - \frac{f(x_n) - f(x_{n-1})}{x_n f(x_n) - x_{n-1} f(x_{n-1})} f(x_n)$

(c) $x_{n+1} = x_n - \frac{f(x_n) - f(x_{n-1})}{x_{n-1} - x_n} f(x_n)$

(d) $x_{n+1} = x_n - \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} f(x_n)$

6. The convergence in bisection method is :

- (a) linear (b) quadratic
 (c) fast (d) slow

7. The rate of convergence of Newton-Raphson Method is :

- (a) Linear (b) Quadratic
 (c) Cubic (d) None of above

8. The order of convergence of Newton Raphson method is :

- (a) 2 (b) 3
 (c) 0 (d) None

9. Newton-Raphson's iterative formula to find the value of $\sqrt{N}$ is :

(a) $x_{n+1} = \frac{1}{2} \left[x_n + \frac{N}{x_n} \right]$ (b) $x_{n+1} = x_n + \frac{N}{x_n}$
 (c) $x_{n+1} = \frac{1}{2} \left[x_n + \frac{\sqrt{N}}{x_n} \right]$ (d) None of these

10. The first approximate of the root of $x^3 - 4x - 9 = 0$ using bisection method is :

- (a) 3.5 (b) 2
 (c) 2.5 (d) None of these

11. If root of $f(x) = 0$ lies in the interval $[a, b]$ then $f(a) \cdot f(b)$ is :

- (a) Greater than zero (b) Less than zero
 (c) Equal to zero (d) None

12. If root of $f(x) = 0$ is a then :

- (a) $f(a) < 0$ (b) $f(b) > 0$
 (c) $f(a) = 0$ (d) None

13. The first approximation of the root of $x^3 - x - 1 = 0$ by using bisection method is :

- (a) 0 (b) 1
 (c) 1.5 (d) None of these

14. For the equation $x^3 - 9x + 1 = 0$, $x_1 = 2$ and $x_2 = 3$ are such that $f(x_1) f(x_2) < 0$, then by false position method an approximation x_n of the root is :

- (a) 2.4 (b) 2.5
 (c) 2.7 (d) 2.9

15. The first approximation of the root of $x^3 - x - 4 = 0$, using the bisection method is :

- (a) 2 (b) 3
 (c) 2.5 (d) 0

16. For the equation $x^3 - 5x - 7 = 0$ using the Regula Falsi Method first approximation root of equation is :
 (a) 2.6 (b) 2.4
 (c) 2.8 (d) 2.5
17. The first approximation of the root of $x^3 - 3x - 5 = 0$ by using Newton Raphson's Method taking $x_0 = 3$ is :
 (a) 2.2 (b) 2.4
 (c) 2.6 (d) 2.8
18. The first approximation of the root of $x^3 - 3x + 1 = 0$ by using Newton-Raphson's Method taking $x_0 = 0.5$ is :
 (a) 0.33 (b) 0.4
 (c) 0.2 (d) 0.44

Answer

- | | | | | |
|--------|--------|--------|--------|--------|
| 1.(c) | 2.(a) | 3.(a) | 4.(a) | 5.(a) |
| 6.(d) | 7.(b) | 8.(a) | 9.(a) | 10.(c) |
| 11.(b) | 12.(c) | 13.(c) | 14.(d) | 15.(c) |
| 16.(b) | 17.(b) | 18.(a) | | |

□□□

Unit-II

Interpolation and Numerical Integration : 1. Lagrange interpolation, 2. Finite difference operators, 3. Interpolation formula using differences- (i) Gregory-Newton Forward Difference Interpolation, (ii) Gregory-Newton Backward Difference Interpolation. 4. Numerical Integration- (i) Newton-Cot's formulae, (ii) Trapezoidal rule, (iii) Simpson's 1/3 Rule, (iv) Simpson's 3/8 Rule, (v) Gauss Integration.

Interpolation

The interpolation is a process of computing intermediate values of a function from a given set of tabular values of a function.

Let $y = f(x)$ be given function

The value of independent variable x are called argument and corresponding value of dependent variable y i.e. $f(x)$ are called entry.

Let

x	:	a	$a + h$	$a + 2h$	$a + 3h$	
$y = f(x)$	:	$f(a)$	$f(a + h)$	$f(a + 2h)$	$f(a + 3h)$	...

with equal interval h , then

Forward difference operator

$$\Delta f(a) = f(a + h) - f(a)$$

is called first forward difference of $f(a)$

$$\Delta f(a + h) = f(a + 2h) - f(a + h)$$

is first forward difference of $f(a + h)$

and
$$\Delta^2 f(a) = \Delta[\Delta f(a)] = \Delta[f(a + h) - f(a)]$$

$$= \Delta f(a + h) - \Delta f(a)$$

$$= \{f(a + 2h) - f(a + h)\} - \{f(a + h) - f(a)\}$$

$$= f(a) = f(a + 2h) - 2f(a + h) + f(a)$$

is called second forward difference of $f(a)$ and in general

$$\Delta^n f(x) = \Delta^{n-1} f(x + h) - \Delta^{n-1} f(x)$$

is n^{th} forward difference of $f(x)$ and Δ is called forward difference operator

Backward difference operator

$$\nabla f(a) = f(a) - f(a - h)$$

is called first backward difference of $f(a)$ and ∇ is called backward difference operator.

and $\nabla f(a-h) = f(a-h) - f(a-2h)$ is first backward difference of $f(a-h)$, and

$$\begin{aligned}\nabla^2 f(a) &= \nabla [\nabla f(a)] = \nabla [f(a) - f(a-h)] \\ &= \nabla f(a) - \nabla f(a-h) \\ &= \{f(a) - f(a-h)\} - \{f(a-h) - f(a-2h)\}\end{aligned}$$

or $\nabla^2 f(a) = f(a) - 2f(a-h) + f(a-2h)$ is second backward difference of $f(a)$ and

$$\nabla^n f(x) = \nabla^{n-1} f(x) - \nabla^{n-1} f(a-h)$$

is n^{th} backward difference of $f(x)$

Forward Difference Table

Argument x	Entry f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
a	f(a)				
a+h	f(a+h)	$f(a+h) - f(a) = A$ (say) $\rightarrow$	$B - A = A_1$ (say) $\rightarrow$	$A_2 - A_1 = B_1$ (say)	$B_2 - B_1 = C_1$ (say)
a+2h	f(a+2h)	$f(a+2h) - f(a+h) = B$ (say)	$C - B = A_2$ (say) $\rightarrow$	$A_3 - A_2 = B_2$ (say)	
a+3h	f(a+3h)	$f(a+3h) - f(a+2h) = C$ (say)	$D - C = A_3$ (say)		
a+4h	f(a+4h)	$f(a+4h) - f(a+3h) = D$ (say)			

Backward Difference Operator

Argument x	Entry f(x)	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
a	f(a)				
a+h	f(a+h)	$f(a+h) - f(a) = A$ (say) $\rightarrow$	$B - A = A_1$ say $\rightarrow$	$A_2 - A_1 = B_1$ say $\rightarrow$	$B_2 - B_1 = C_1$ (say)
a+2h	f(a+2h)	$f(a+2h) - f(a+h) = B$ (say) $\rightarrow$	$C - B = A_2$ (say) $\rightarrow$	$A_3 - A_2 = B_2$ (say)	
a+3h	f(a+3h)	$f(a+3h) - f(a+2h) = C$ $\rightarrow$	$D - C = A_3$ (say)		
a+4h	f(a+4h)	$f(a+4h) - f(a+3h) = D$ $\rightarrow$			

Shifting or Enlargement or Translation Operator

$$Ef(x) = f(x+h)$$

is called first shifting operator

$$E^2 f(x) = f(x+2h)$$

$$E^n f(x) = f(x+nh)$$

and $E^{-1} f(x) = f(x-h)$

$$E^{-2} f(x) = f(x-2h)$$

$$\dots \text{ and } E^{-n} f(x) = f(x-nh)$$

Average Operator μ

$$\mu f(x) = \frac{1}{2} \left[f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right) \right]$$

Differential Operator D

$$Df(x) = \frac{d}{dx} f(x)$$

Central Difference Operator

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$$

Q.1. Prove that :

$$(1) \Delta f(x) = (E - 1) f(x) \quad (2) \Delta^n f(x) = (E - 1)^n f(x)$$

$$(3) E \nabla = \nabla E = \Delta \quad (4) (\Delta - \nabla) = \Delta \nabla$$

$$(5) (1 + \Delta)(1 - \nabla) = 1$$

Sol. : We know that by definition

$$\Delta f(x) = f(x+h) - f(x) \quad \dots (1)$$

$$\text{and } Ef(x) = f(x+h) \quad \dots (2) \text{ and } \nabla f(x) = f(x) - f(x-h) \quad \dots (3)$$

$$(1) \Delta f(x) = (E - 1) f(x)$$

L.H.S.

$$= \Delta f(x) = f(x+h) - f(x) \quad \text{By (1)}$$

$$= E f(x) - f(x) \quad \text{By (2)}$$

$$= (E - 1) f(x)$$

$$= \text{R.H.S.}$$

$$(2) \Delta^n f(x) = (E - 1)^n f(x)$$

$$\text{By (1)} \quad \Delta = (E - 1)$$

$$\Rightarrow \Delta^n = (E - 1)^n$$

$$\Delta^n f(x) = (E - 1)^n f(x)$$

$$(3) E \nabla = \nabla E = \Delta$$

$$\therefore E \nabla f(x)$$

$$\begin{aligned}
 &= E[f(x) - f(x-h)] \\
 &= Ef(x) - Ef(x-h) \\
 &= f(x+h) - f(x) \\
 &= \Delta f(x) \\
 &= E \nabla f(x) = \Delta f(x) \text{ or } \nabla = \Delta
 \end{aligned}$$

Now $\nabla E f(x)$

$$\begin{aligned}
 &= \nabla [f(x+h)] \\
 &= f(x+h) - f(x) \\
 &= \Delta f(x)
 \end{aligned}$$

$$\Rightarrow \nabla E f(x) = \Delta f(x) \Rightarrow \nabla E = \Delta$$

$$(4) \quad (\Delta - \nabla) = \Delta \nabla$$

 $\therefore$ consider $\Delta \nabla f(x)$

$$\begin{aligned}
 &= \Delta[f(x) - f(x-h)] \\
 &= \Delta f(x) - [f(x) - f(x-h)] \\
 &= \Delta f(x) - \nabla f(x) \\
 &= [\Delta - \nabla] f(x)
 \end{aligned}$$

$$\text{or } \Delta \nabla f(x) = [\Delta - \nabla] f(x)$$

$$\text{or } \Delta - \nabla = \Delta \nabla$$

$$(5) \quad (1 + \Delta)(1 - \nabla) = 1$$

consider $(1 + \Delta)(1 - \Delta) f(x)$

$$\begin{aligned}
 &= [1 + \Delta][f(x) - \nabla f(x)] \\
 &= (1 + \Delta)[f(x) - \{f(x) - f(x-h)\}] \\
 &= E[f(x-h)] \\
 &= f(x)
 \end{aligned}$$

By $1 + \Delta = E$

By (2)

$$\Rightarrow (1 + \Delta)[1 - \nabla] f(x) = f(x)$$

$$\text{or } [1 + \Delta][1 - \nabla] = 1$$

Q.2. Prove that : (1) $\Delta \log f(x) = \log \left(1 + \frac{\Delta f(x)}{f(x)} \right)$

$$(2) e^x = \left(\frac{\Delta^2}{E} \right) e^x \frac{Ee^x}{\Delta^2 e^x}$$

Sol : (1) L.H.S.

$$\begin{aligned}
 &= \Delta \log f(x) \\
 &= \log \left[\frac{f(x+h)}{f(x)} \right]
 \end{aligned}$$

$$= \log \left[\frac{Ef(x)}{f(x)} \right]$$

$$\therefore Ef(x) = f(x+h)$$

$$= \log \left[\frac{(1 + \Delta)f(x)}{f(x)} \right]$$

$$\therefore E = 1 + \Delta$$

$$= \log \left[\frac{f(x) + \Delta f(x)}{f(x)} \right]$$

$$= \log \left[\frac{f(x)}{f(x)} + \frac{\Delta f(x)}{f(x)} \right]$$

$$= \log \left[1 + \frac{\Delta f(x)}{f(x)} \right]$$

$$= \text{R.H.S.}$$

(2) R.H.S.

$$= \left(\frac{\Delta^2}{E} \right) e^x \frac{Ee^x}{\Delta^2 e^x}$$

$$= (\Delta^2 \cdot E^{-1}) e^x \frac{Ee^x}{\Delta^2 e^x}$$

$$= \Delta^2 [E^{-1} e^x] \frac{Ee^x}{\Delta^2 e^x}$$

$$= \Delta^2 [e^{x-h}] \frac{e^{x+h}}{\Delta^2 e^x}$$

$$\therefore Ef(x) = f(x+h)$$

$$= e^{-h} \Delta^2 e^x \cdot \frac{e^x \cdot e^h}{\Delta^2 e^x}$$

$$= e^x = \text{L.H.S.}$$

Q.3. Evaluate $\Delta^n \frac{1}{x}$ with $h = 1$

$$\text{Sol : } \therefore \Delta \frac{1}{x} = \frac{1}{x+1} - \frac{1}{x} \text{ with } h = 1$$

$$\Delta \frac{1}{x} = \frac{-1}{x(x+1)}$$

(1)

$$\Delta^2 \frac{1}{x} = \Delta \left[\Delta \frac{1}{x} \right] = \Delta \left[\frac{-1}{x(x+1)} \right]$$

By (1)

$$= -\left[\frac{1}{(x+1)(x+2)} - \frac{1}{x(x+1)}\right]$$

$$\Delta^2 \frac{1}{x} = \frac{(-1)^2 2!}{x(x+1)(x+2)}$$

$$\text{Now } \Delta^3 \frac{1}{x} = \Delta \left[\Delta^2 \frac{1}{x} \right] = \Delta \left[\frac{(-1)^2 2!}{x(x+1)(x+2)} \right]$$

$$= (-1)^2 2! \left[\frac{1}{(x+1)(x+2)(x+3)} - \frac{1}{x(x+1)(x+2)} \right] \text{ By (2)}$$

$$\Delta^3 \frac{1}{x} = \frac{(-1)^3 3!}{x(x+1)(x+2)(x+3)}$$

By (1), (2), (3) and so on

$$\Delta^n \frac{1}{x} = \frac{(-1)^n n!}{x(x+1)(x+2)\dots(x+n)}$$

Q. 4. Estimate the missing term in the following table:

x:	0	1	2	3	4
y:	1	3	9	?	81

Sol.: Here the number of known value is 4 then

$$\Delta^4 f(x) = 0$$

$$\text{or } (E-1)^4 f(x) = 0$$

$$\Rightarrow [E^4 - 4C_1 E^3 + 4C_2 E^2 - 4C_3 E + 1] f(x) = 0$$

$$\Rightarrow E^4 f(x) - 4E^3 f(x) + \frac{4.3}{2.1} E^2 f(x) - \frac{4.3.2}{3.2.1} E f(x) + f(x) = 0$$

Here $h = 1$

$$f(x+4.1) - 4f(x+3.1) + 6f(x+2.1) - 4f(x+1) + f(x) = 0$$

Put the first value of x i.e. $x = 0$

$$\Rightarrow f(0+4) - 4f(0+3) + 6f(0+2) - 4f(0+1) + f(0) = 0$$

$$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$$

$$81 - 4f(3) + 6(9) - 4(3) + (1) = 0$$

$$124 - 4f(3) = 0 \Rightarrow f(3) = \frac{124}{4} = 31$$

$$\therefore f(3) = 31$$

Q.5. Find the missing values in the following table:

x:	0	5	10	15	20	25
y:	6	10	-	17	-	31

Sol.: Here, the number of known values is 4 then $\Delta^4 f(x) = 0$

$$(E-1)^4 f(x) = 0$$

$$\Rightarrow [E^4 - 4C_1 E^3 + 4C_2 E^2 - 4C_3 E + 1] f(x) = 0$$

$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$$

$$\text{Here } h = 5$$

$$f(x+4(5)) - 4f(x+3(5)) + 6f(x+2(5)) - 4f(x+1(5)) + f(x) = 0$$

$$f(x+20) - 4f(x+15) + 6f(x+10) - 4f(x+5) + f(x) = 0 \quad \dots(*)$$

Put the first value of x i.e. $x = 0$

$$f(0+20) - 4f(0+15) + 6f(0+10) - 4f(0+5) + f(0) = 0$$

$$f(20) - 4f(15) + 6f(10) - 4f(5) + f(0) = 0$$

$$f(20) - 4(17) + 6f(10) - 4(10) + 6 = 0$$

$$f(20) + 6f(10) = 102 \quad \dots(1)$$

Now put the second value of x is

$$x = 5 \text{ in } (*)$$

$$f(5+20) - 4f(5+15) + 6f(5+10) - 4f(5+5) + f(5) = 0$$

$$31 - 4f(20) + 6(17) - 4f(10) + 10 = 0$$

$$4f(20) + 4f(10) = 143 \quad \dots(2)$$

Solving (1) and (2), we obtain

$$f(10) = 13.25$$

$$\text{and } f(20) = 22.5$$

Q.6. Find the missing value of the table given below:

Year :	1917	1918	1919	1920	1921
Export :	443	384	-	397	467
(in tons)					

Sol.: Here, the number of known values is 4 then

$$\Delta^4 f(x) = 0$$

$$\text{or } (E-1)^4 f(x) = 0$$

$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$$

$$\therefore h = 1$$

$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0$$

Put the first value of x i.e. $x = 1917$

$$f(1917+4) - 4f(1917+3) + 6f(1917+2) - 4f(1917+1) + f(1917) = 0$$

$$f(1921) - 4f(1920) + 6f(1919) - 4f(1918) + f(1917) = 0$$

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$$467 - 4(397) + 4f(1919) - 4(384) + 443 = 0$$

$$4f(1919) = 2214$$

$$\Rightarrow f(1919) = 369$$

Ans.

Newton's Forward Difference Interpolation Formula

Given that

$$x : a \quad a+h \quad a+2h \quad a+3h \quad \dots$$

$$f(x) : f(a) \quad f(a+h) \quad f(a+2h) \quad f(a+3h) \quad \dots$$

then for given value of x ,

$$f(x) = f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \dots$$

Newton's Backward Difference Interpolation Formula

Given that

$$x : a \quad a+h \quad a+2h \quad a+3h \quad \dots$$

$$f(x) : f(a) \quad f(a+h) \quad f(a+2h) \quad f(a+3h) \quad \dots$$

then for given value of x ,

$$f(x) = f(a+nh+uh) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u-1)}{2!} \nabla^2 f(a+nh) + \frac{u(u+1)(u+2)}{3!} \nabla^3 f(a+nh) + \dots$$

Q.7. Estimate the sale for 1966 using the following table :

Year	: 1931	1941	1951	1961	1971	1981
Sale in	: 12	15	20	27	39	52
	thousand					

Sol. : Here $h = 10$, $x = 1966$, $f(x) = f(1966) = ?$ then by Newton's Forward Interpolation Formula :

$$f(x) = f(a+uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \dots (*)$$

$$\therefore x = 1966 \text{ \& } a = 1931 \text{ and } x = a + uh \Rightarrow 1966 = 1931 + u(10)$$

$$u = \frac{35}{10} = 3.5$$

Its forward difference table :

Year	Sale $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	
1931	12					
	$\rightarrow$	03				
1941	15	$\rightarrow$	2			
	$\rightarrow$	05	$\rightarrow$	0		
1951	20	$\rightarrow$	2	$\rightarrow$	3	
	$\rightarrow$	07	$\rightarrow$	3	$\rightarrow$	-10
1961	27	$\rightarrow$	5	$\rightarrow$	-7	
	$\rightarrow$	12	$\rightarrow$	-4		
		$\rightarrow$	1			
1971	39	$\rightarrow$	13			
1981	52					

By (*)

$$\begin{aligned} f(1966) &= 12 + 3.5(3) + \frac{(3.5)(3.5-1)}{2!} (2) \\ &+ \frac{3.5(3.5-1)(3.5-2)}{3!} (0) + \frac{3.5(3.5-1)(3.5-2)(3.5-3)}{4!} \\ &(3) + \frac{3.5(3.5-1)(3.5-2)(3.5-3)(3.5-4)}{5!} - (-10) \\ &= 12 + 10.5 + 8.75 + 0.8203 + 0.2734 \end{aligned}$$

$$\Rightarrow f(1966) = 32.3437$$

 $\therefore$ Sale for year 1966 is 32.3437 thousandsQ.8. Given that $\sin 45^\circ = 0.7071$, $\sin 50^\circ = 0.7660$, $\sin 55^\circ = 0.8192$, $\sin 60^\circ = 0.8660$, find $\sin 52^\circ$, by using Newton's Forward Difference Interpolation Formula.

Givent that

Sol: x	:	45°	50°	55°	60°
f(x)	:	0.7071	0.7660	0.8192	0.8660

Here $a = 45^\circ$, $h = 5^\circ$, $x = 52^\circ$, $f(x) = ?$

For backward difference table

s	f(s)	$\nabla f(s)$	$\nabla^2 f(s)$	$\nabla^3 f(s)$	$\nabla^4 f(s)$
1921	101				
1922	99	-2			
1923	97	-2	-2		
1924	95	-2	-2	-2	
1925	93	-2	-2	-2	-2
1926	91	-2	-2	-2	-2
1927	89	-2	-2	-2	-2

By Newton's Backward Difference Formula

$$f(x) = f(a + sh) + u \nabla f(a + sh) + \frac{u(u+1)}{2!} \nabla^2 f(a + sh) + \frac{u(u+1)(u+2)}{3!} \nabla^3 f(a + sh) + \dots$$

$$\begin{aligned} f(1926) &= 101 + (-0.6)(-2) + \frac{(-0.6)(-0.6+1)}{2!} (-2) \\ &\quad + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!} (-2) \\ &\quad + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!} (-2) \end{aligned}$$

$$\begin{aligned} f(1926) &= 101 - 1.2 + 0.48 - 0.006 + 0.006 \\ f(1926) &= 99.288 \text{ thousands} \end{aligned}$$

Q 19. In an examination the number of candidates who secured marks between certain limits were as follows:

Marks	0-10	10-20	20-30	30-40	40-50	50-60
No. of candidates	41	62	68	59	37	17

Estimate the number of candidates getting marks less than 70.

Forward difference table

Marks less than	No. of candidates	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	41			
10	103	62		
20	171	68	6	
30	239	71	3	-3
40	310	71	0	0
50	347	37	-34	34
60	384	37	0	0

39	103	→	03		
		→	65	→	-18
59	168	→		→	-15
		→	50	→	-18
79	218	→		→	-33
		→	17		
99	235				

Here

$$a = 19, h = 20, x = 70$$

$$x = a + uh \Rightarrow u = \frac{x-a}{h} = \frac{70-19}{20} = 2.55$$

By Newton's formula

$$f(x) = f(a + uh) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \dots$$

$$= 41 + 2.55(62) + \frac{2.55(2.55-1)}{2} (03) + \frac{2.55(2.55-1)(2.55-2)}{3!} (1-18) + 0 + \dots$$

$$= 41 + 158.10 + 5.92 - 6.52$$

$$\Rightarrow f(70) = 198.50 = 198 \text{ approximate}$$

Q.11. Find the polynomial which fits the data in the following table :

Or

Find the value of $f(x)$ using following data :

x	:	0	1	2	3
y	:	1	0	1	10

also find $f(2.5)$.Sol : Here $a = 0, x = x, h = 1$ and $x = a + hu$

$$\Rightarrow u = \frac{x-a}{h} = \frac{x-0}{1} = x$$

Its formula difference table :

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
1	0	-1		
2	1	1	2	
3	10	9	8	6

By Newton's Formula

$$f(x) = f(a + hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \dots$$

$$\Rightarrow f(x) = 1 + x(-1) + \frac{x(x-1)2}{2} + \frac{x(x-1)(x-2)}{3 \cdot 2 \cdot 1} 6 + 0$$

$$= 1 - x + x^2 - x + x^3 - 3x^2 + 2x$$

$$f(x) = x^3 - 2x^2 + 1$$

$$\text{By (*) } f(2.5) = (2.5)^3 - 2(2.5)^2 + 1 = 4.125$$

Ans.

Q.12. Find a polynomial which takes the values :

x	:	0	1	2	3	4	5	6	7
y	:	1	2	4	7	11	16	22	29

Sol : Its forward differences table :

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
1	2	1		
2	4	2	1	
3	7	3	1	
4	11	4	1	
5	16	5	1	
6	22	6	1	
7	29	7	1	

Here $a = 0, x = x, h = 1$ and $x = a + hu$

$$\Rightarrow u = \frac{x-a}{h} = \frac{x-0}{1} = x$$

By Newton's forward difference formula

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!} \Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!} \Delta^3 f(a) + \dots$$

$$\Rightarrow f(x) = 1 + x \cdot 1 + \frac{x(x-1)}{2!} \cdot 1 + 0 + \dots$$

$$f(x) = \frac{x^2}{2} + \frac{x}{2} + 1$$

Divided Difference Operator

Given that (unequal interval)

$$\begin{array}{ccccccc} x : & x_0 & x_1 & x_2 & x_3 & \dots & x_n \\ y = f(x) : & f(x_0) & f(x_1) & f(x_2) & f(x_3) & \dots & f(x_n) \end{array}$$

then

$$\Delta_{x_1} f(x_0) = f(x_0, x_1) = [x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

be first divided difference of $f(x_0)$ and Δ is called divided difference operator

$$\Delta_{x_2} f(x_1) = f(x_1, x_2) = [x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

be first divided difference operator and

$$\Delta_{x_1 x_2}^2 f(x_0) = f(x_0, x_1, x_2) = [x_0, x_1, x_2] = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_0}$$

be second divided difference operator

$$\Delta_{x_2 x_3}^2 f(x_1) = f(x_1, x_2, x_3) = [x_1, x_2, x_3] = \frac{\Delta f(x_2) - \Delta f(x_1)}{x_3 - x_1}$$

be second divided difference operator $f(x_1)$

$$\Delta_{x_1 x_2 x_3}^3 f(x_0) = f(x_0, x_1, x_2, x_3) = [x_0, x_1, x_2, x_3] = \frac{\Delta^2 f(x_2) - \Delta^2 f(x_1)}{x_3 - x_0}$$

be third divided difference operator $f(x)$ and so

$$\Delta_{x_1 x_2 \dots x_n}^n f(x) = f(x_0, x_1, x_2, \dots, x_n) = [x_0, x_1, x_2, \dots, x_n]$$

$$= \frac{\Delta^{n-1} f(x_1) - \Delta^{n-1} f(x_0)}{x_n - x_0}$$

is n^{th} divided difference operator of $f(x)$ **Newton's Divided Difference Interpolation Formula**

Given that :

$$\begin{array}{ccccccc} x & : & x_0 & x_1 & x_2 & \dots & x_n \\ y = f(x) & : & f(x_0) & f(x_1) & f(x_2) & \dots & f(x_n) \end{array}$$

then for given value of x ,

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2) \Delta^3 f(x_0) + \dots + (x - x_0)(x - x_1) \dots (x - x_{n-1}) \Delta^n f(x_0)$$

Lagrange's Interpolation Formula for unequal intervals

Given that :

$$\begin{array}{ccccccc} x & : & x_0 & x_1 & x_2 & \dots & x_n \\ y = f(x) & : & f(x_0) & f(x_1) & f(x_2) & \dots & f(x_n) \end{array}$$

then for given value of x ,

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \dots (x_0 - x_n)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)(x - x_3) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n)} f(x_1) +$$

$$\frac{(x - x_0)(x - x_1)(x - x_3) \dots (x - x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_2) \dots (x_n - x_{n-1})} f(x_n)$$

Q.13. (1) Prove that $\Delta_{yz}^2 x^2$ is independent of x, y, z .(2) Prove that $\Delta_{yz}^2 x^3 = x + y + z$ (3) Prove that $\Delta_{bcd}^3 = \frac{1}{a} = -\frac{1}{abcd}$ **Sol :** (1) $\Delta_{yz}^2 x^2$

$$\text{Here } f(x) = x^2 \quad \dots(1)$$

$$\Delta x^2 = \Delta f(x) = \frac{f(y) - f(x)}{y - x}$$

$$= \frac{y^2 - x^2}{y - x} = y + x$$

$$\Rightarrow \Delta x^2 = y + x \quad \dots(2)$$

$$\text{similarly } \Delta y^2 = z + y \quad \dots(3)$$

$$\begin{aligned} \therefore \Delta^2 x^2 &= \frac{\Delta y^2 - \Delta x^2}{z - x} \\ &= \frac{(z + y) - (y + x)}{z - x} = 1 \end{aligned}$$

$$\Delta x^2 = 1$$

which is independent of x, y, z (2) $\Delta^2 x^3$, here $f(x) = x^3$... (1)

$$\therefore \Delta x^3 = \Delta f(x) = \frac{f(y) - f(x)}{y - x} = \frac{y^3 - x^3}{y - x}$$

$$\Delta f(x) = y^2 + yx + x^2 \quad \dots(2)$$

$$\text{similarly } \Delta f(y) = z^2 + zy + y^2 \quad \dots(3)$$

$$\therefore \Delta^2 x^3 = \Delta^2 f(x) = \frac{\Delta f(y) - \Delta f(x)}{z - x} = \frac{(z^2 + zy + y^2) - (y^2 + yx + x^2)}{z - x}$$

$$= \frac{z^2 - x^2 + zy - yx}{z - x} = \frac{(z-x)(z+x) + y(z-x)}{(z-x)}$$

$$\Rightarrow \Delta^2 x^3 = x + y + z$$

$$(3) \Delta^2 \frac{1}{a}, \text{ Here } f(a) = \frac{1}{a}$$

$$\Delta f(a) = \frac{1}{a} = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b} - \frac{1}{a}}{b - a}$$

$$\Rightarrow \Delta \frac{1}{a} = -\frac{1}{ab}$$

$$\text{similarly } \Delta \frac{1}{b} = -\frac{1}{bc}$$

$$\Delta \frac{1}{c} = -\frac{1}{cd}$$

$$\Delta^2 \frac{1}{a} = \Delta f(a) = f(a, b, c) = \frac{\Delta f(b) - \Delta f(a)}{c - a}$$

$$= \frac{-\frac{1}{bc} + \frac{1}{ab}}{c - a} = \frac{\frac{abc}{c - a}}$$

$$\Delta^2 \frac{1}{a} = \frac{1}{abc}$$

$$\text{and } \Delta \frac{1}{b} = \frac{1}{bcd}$$

$$\Delta^3 \frac{1}{a} = \Delta^2 f(a) = f(a, b, c, d) = \frac{\Delta^2 \frac{1}{b} - \Delta^2 \frac{1}{a}}{d - a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d - a} = \frac{\frac{abcd}{d - a}}{d - a}$$

$$\Delta^3 \frac{1}{a} = -\frac{1}{abcd}$$

Q.14. If $f(x) = \frac{1}{x}$, then prove that

$$\Delta^n f(x) = f(x_0, x_1, x_2, \dots, x_n) = (-1)^n \frac{1}{x_0 x_1 x_2 \dots x_n}$$

Sol: Given that $f(x) = \frac{1}{x}$

$$\Delta f(x_0) = f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\frac{1}{x_1} - \frac{1}{x_0}}{x_1 - x_0}$$

$$\Delta f(x_0) = \frac{1}{x_0 x_1} \quad \dots(1)$$

$$\text{similarly } \Delta f(x_1) = \frac{1}{x_1 x_2} \quad \dots(2)$$

then

$$\Delta^2 f(x_0) = f(x_0, x_1, x_2) = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_0} = \frac{-\frac{1}{x_1 x_2} - \frac{1}{x_0 x_1}}{x_2 - x_0}$$

$$\Delta^2 f(x_0) = (-1)^2 \frac{1}{x_0 x_1 x_2} \quad \dots(3)$$

$$\text{Similarly } \Delta^2 f(x_1) = (-1)^2 \frac{1}{x_1 x_2 x_3} \quad \dots(4)$$

then

$$\Delta^3 f(x_0) = f(x_0, x_1, x_2, x_3) = \frac{\Delta^2 f(x_1) - \Delta^2 f(x_0)}{x_3 - x_0}$$

$$\Rightarrow \Delta^3 f(x) = \frac{\frac{(-1)^2}{x_1 x_2 x_3} - \frac{(-1)^2}{x_0 x_1 x_2}}{x_3 - x_0}$$

$$\Delta^3 f(x_0) = (-1)^3 \frac{1}{x_0 x_1 x_2 x_3}$$

and so on

$$\Delta^n f(x_0) = f(x_0, x_1, x_2, \dots, x_n) = (-1)^n \frac{1}{x_0 x_1 x_2 \dots x_n}$$

Q.15. Find $f(a, b, c, d)$ for $f(x) = \frac{1}{x}$

$$\text{Sol: Given that } f(x) = \frac{1}{x} \quad \dots(1)$$

$$\text{then } \Delta_1 f(a) = f(a, b) = \frac{f(b) - f(a)}{b - a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b - a}$$

$$= \frac{\frac{a^2 - b^2}{a^2 b^2}}{b - a} = \frac{(a - b)(a + b)}{a^2 b^2 (b - a)}$$

$$\Delta_1 f(a) = f(a, b) = \frac{(a + b)}{a^2 b^2}$$

Similarly,

$$f(b, c) = \frac{(b + c)}{b^2 c^2}$$

$$f(c, d) = \frac{(c + d)}{c^2 d^2}$$

Now

$$\Delta_{bc}^2 f(a) = f(a, b, c) = \frac{f(b, c) - f(a, b)}{c - a}$$

$$= \frac{-\left(\frac{(b + c)}{b^2 c^2}\right) + \left(\frac{(a + b)}{a^2 b^2}\right)}{c - a}$$

$$= \frac{-a^2 b - a^2 c + c^2 a + bc^2}{a^2 b^2 c^2 (c - a)} = \frac{b(c^2 - a^2) + ca(c - a)}{a^2 b^2 c^2 (c - a)}$$

$$= f(a, b, c) = \frac{ab + bc + ca}{a^2 b^2 c^2} \dots (4)$$

$$\text{similarly } f(b, c, d) = \frac{bc + cd + db}{b^2 c^2 d^2} \dots (5)$$

$$\text{Now } \Delta_{bcd}^3 f(a) = f(a, b, c, d) = \frac{\Delta^2 f(a) - \Delta^2 f(b)}{d - a}$$

$$= \frac{\left[\frac{bc + cd + db}{b^2 c^2 d^2}\right] - \left[\frac{ab + bc + ca}{a^2 b^2 c^2}\right]}{d - a} \text{ By (4) \& (5)}$$

$$= \frac{a^2 bc + a^2 ed + a^2 db - d^2 ab - d^2 bc - d^2 ca}{a^2 b^2 c^2 d^2 (d - a)}$$

$$= \frac{bc + (a^2 - d^2) + acd(a - d) + abd(a - d)}{a^2 b^2 c^2 d^2 (d - a)}$$

$$= \frac{(d - a)(bca + bcd + acd + abd)}{a^2 b^2 c^2 d^2 (d - a)}$$

$$\Rightarrow f(a, b, c, d) = \left[\frac{abc + bcd + cda + dab}{a^2 b^2 c^2 d^2} \right]$$

Q.16. Evaluate $f(9)$, using Newton's divided difference formula, from following data :

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Sol: Let

	x_0	x_1	x_2	x_3	x_4
x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2) \Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3) \Delta^4 f(x_0) + \dots (*)$$

Here $x = 9$, $f(x) = f(9) = ?$

Its divided difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150	$\frac{392 - 150}{7 - 5} = 121$			
	$\rightarrow$	$\rightarrow$	$\frac{265 - 121}{11 - 5} = 24$		
7	392		$\rightarrow$	$\frac{32 - 24}{13 - 5} = 1$	
	$\rightarrow$	$\frac{1452 - 392}{11 - 7} = 265$			
	$\rightarrow$	$\rightarrow$	$\frac{457 - 265}{13 - 7} = 32$		
11	1452		$\rightarrow$	$\frac{42 - 32}{17 - 7} = 1$	
	$\rightarrow$	$\frac{2366 - 1452}{13 - 11} = 457$			
	$\rightarrow$	$\rightarrow$	$\frac{709 - 457}{17 - 13} = 42$		
13	2366		$\rightarrow$	$\frac{5202 - 2366}{17 - 13} = 709$	
	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\frac{1 - 1}{17 - 5} = 0$
17	5202				

Now by (*)

$$= f(9) = 150 + (9 - 5) 121 + (9 - 5)(9 - 7)(9 - 11) 1 + 0$$

$$= 150 + 484 + 192 - 16$$

$$= f(9) = 810$$

Q.17. Given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find $\log_{10} 656$ using Newton's divided difference interpolation formula.

Sol: Let

	x_0	x_1	x_2	x_3
x	654	658	659	661
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Here $x = 656$, $\log x = f(x) = \log 656 = f(656) = ?$

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + \dots \quad (*)$$

Its divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156	$\frac{2.8182 - 2.8156}{658 - 654} = 0.00065$	$\frac{0.0007 - 0.00065}{659 - 654} = 0.00001$	$\frac{0.00017 - 0.00001}{661 - 654} = -0.00004$
658	2.8182	$\frac{2.8189 - 2.8182}{659 - 658} = 0.00070$	$\frac{0.00065 - 0.0007}{661 - 659} = -0.00004$	
659	2.8189	$\frac{2.8202 - 2.8189}{661 - 659} = 0.00065$		
661	2.8202			

Now by (*)

$$\begin{aligned} f(x) &= \log 656 = 2.8156 + (656 - 654)(0.00065) + (656 - 654)(656 - 658)(0.00001) + (656 - 654)(656 - 659)(-0.000004) \\ &= 2.8156 + 0.0013 - 0.00004 - 0.000048 \\ &= 2.816812 \end{aligned}$$

Q.18. Find the equation $y = f(x)$ of least degree and passing through the points $(-1, -21)$, $(1, 15)$, $(2, 12)$, $(3, 3)$. Find also y at $x = 0$

Sol: Given that the points are

	x_0	x_1	x_2	x_3
x	-1	1	2	3
y	-21	15	12	3
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

By Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2) \Delta^3 f(x_0) + \dots \quad (*)$$

Its divided difference table

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-21	$\frac{15 + 21}{1 + 1} = 18$	$\frac{-3 - 18}{2 + 1} = -7$	$\frac{-3 + 7}{3 + 1} = 1$
1	15	$\frac{12 + 15}{2 - 1} = -3$	$\frac{-9 + 3}{3 - 1} = -3$	
2	12	$\frac{3 - 12}{3 - 2} = -9$		
3	3			

By (*)

$$\begin{aligned} f(x) &= -21 + (x + 1)18 + (x + 1)(x - 1)(-7) + (x + 1)(x - 1)(x - 2)1 \\ &= -21 + 18x + 18 - 7x^2 + 7 + x^3 - 2x^2 - x + 2 \\ f(x) &= x^3 - 9x^2 + 17x + 6 \end{aligned}$$

Q.19. Using Newton's divided difference formula. Find the values of $f(2)$, $f(8)$ and $f(15)$ given the following data:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

Sol: Given that

	x_0	x_1	x_2	x_3	x_4	x_5
x	4	5	7	10	11	13
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$
$f(x)$	48	100	294	900	1210	2028

By Newton's divided difference formula:

$$f(x) = f(x_0) + (x - x_0) \Delta f(x_0) + (x - x_0)(x - x_1) \Delta^2 f(x_0) + \dots \quad (*)$$

Its divided difference table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
4	48	$\frac{100-48}{5-4} = 52$	$\frac{97-52}{7-4} = 15$	$\frac{21-15}{10-4} = 1$	$\frac{1-1}{11-4} = 0$	
5	100	$\frac{294-100}{7-5} = 97$	$\frac{202-97}{10-5} = 21$	$\frac{27-21}{11-5} = 1$	$\frac{1-1}{13-5} = 0$	
7	294	$\frac{900-294}{10-7} = 202$	$\frac{310-202}{11-7} = 27$	$\frac{33-27}{13-7} = 1$		
10	900	$\frac{1210-900}{11-10} = 310$	$\frac{409-310}{13-10} = 33$			
11	1210	$\frac{2028-1210}{13-11} = 409$				
13	2028					

∴ By (*)

$$f(x) = 48 + (x-4)52 + (x-4)(x-5)15 + (x-4)(x-5)(x-7)1 + 0 + 0 \dots$$

$$f(x) = (x-4)(x-5)(x-7) + 15(x-4)(x-5) + 52(x-4) + 48 \dots (4)$$

By (A), Put $x = 2$

$$f(2) = (2-4)(2-5)(2-7) + 15(2-4)(2-5) + 52(2-4) + 48$$

$$= -30 + 90 - 104 + 48$$

$$\Rightarrow f(2) = 4$$

Ans.

By (*)

$$f(8) = (8-4)(8-5)(8-7) + 15(8-4)(8-5) + 52(8-4) + 48$$

$$\Rightarrow f(8) = 448$$

and By (*)

$$f(15) = (15-4)(15-5)(15-7) + 15(15-4)(15-5)$$

$$+ 52(15-4) + 48$$

$$\Rightarrow f(15) = 3150$$

Q.20. Using Lagrange's interpolation formula find $y(10)$ from the following table :

x :	5	6	9	11
y :	12	13	14	16

Sol. : Let $x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$
 $f(x) = y : 12, 13, 14, 16$
 $f(x_0) \quad f(x_1) \quad f(x_2) \quad f(x_3)$

By Lagrange's interpolation formula :

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

Put $x = 10$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \cdot 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \cdot 13$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \cdot 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \cdot 16$$

$$= \frac{4(1)(-1)}{(-1)(-4)(-6)} 12 + \frac{5(1)(-1)}{1(-3)(-5)} 13 + \frac{5(4)(-1)}{4(3)(-2)} 14 + \frac{5(4)(1)}{6(5)(2)} 16$$

$$= 2 - 4.33333 + 11.66666 + 5.33333$$

$$f(10) = 14.66666$$

Q.21. Calculate $\log_{10} 656$ by Lagrange's method, given $\log_{10} 654 = 2.81569$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$ and $\log_{10} 661 = 2.8202$

Sol. : Given that

	x_0	x_1	x_2	x_3
x :	654	658	659	661
$\log x = f(x)$:	2.8156	2.8182	2.8189	2.8202
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Here $x = 656$ & $f(x) = f(656) = \log_{10} 656 = ?$

By Lagrange's interpolation formula :

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\begin{aligned}
& \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) + \\
& \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\
\Rightarrow f(656) = \log_{10} 656 &= \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \cdot 2.8156 \\
&+ \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \cdot 2.8182 \\
&+ \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \cdot 2.8189 + \\
&\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \cdot 2.8202 \\
\Rightarrow \log_{10} 656 &= \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \cdot 2.8156 + \frac{2(-3)(-5)}{4(-1)(-3)} \cdot 2.8182 \\
&+ \frac{(2)(-2)(-5)}{5(1)(-2)} \cdot 2.8189 + \frac{(2)(-2)(-3)}{7(3)(2)} \cdot 2.8202 \\
&= 0.6033 + 7.0455 - 5.6378 + 0.8058 \\
\Rightarrow \log_{10} 656 &= 2.8168
\end{aligned}$$

Q.22. Given that $\log_{10} 300 = 2.4771$, $\log_{10} 304 = 2.4829$, $\log_{10} 305 = 2.4843$, $\log_{10} 307 = 2.4871$. Find $\log_{10} 301$

Sol. Given that

	x_0	x_1	x_2	x_3
x	300	304	305	307
$\log_{10} x$	2.4771	2.4829	2.4843	2.4871
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Here $x = 301$ and $f(x) = \log_{10} 301 = ?$

By Lagrange interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) +$$

$$\begin{aligned}
& \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\
& + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)
\end{aligned}$$

Put the values

$$\begin{aligned}
f(301) = \log_{10} 301 &= \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)} \cdot 2.4771 + \\
&\frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)} \cdot 2.4829 + \\
&\frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)} \cdot 2.4843 + \\
&\frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} \cdot 2.4871 \\
\log_{10} 301 &= \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} \cdot 2.4771 + \frac{1(-4)(-6)}{4(-1)(-3)} \cdot 2.4829 + \\
&\frac{1(-3)(-6)}{5(1)(-2)} \cdot 2.4843 + \frac{1(-3)(-4)}{7(3)(2)} \cdot 2.4871 \\
&= 1.27393 + 4.9658 - 4.4717 + 0.7106 \\
\Rightarrow \log_{10} 301 &= 2.4786
\end{aligned}$$

Ans.

Q.23. Find the polynomial which takes the values given below :

x	:	0	1	2	4
y	:	1	1	2	5

Sol : Given that :

	x_0	x_1	x_2	x_3	
x :	0	1	2	4	
$f(x)$:	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	5
	1	1	2		

By Lagrange-Interpolation formula

$$\begin{aligned}
f(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \\
&\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\
&+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \\
&\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)
\end{aligned}$$

Put the values

$$\begin{aligned}
 f(x) &= \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)} \cdot 1 + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)} \cdot 1 \\
 &\quad + \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)} \cdot 2 + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)} \cdot 5 \\
 &= \frac{x^3 - 7x^2 + 14x - 8}{8} + \frac{x^3 - 6x^2 + 8x}{3} + \left(\frac{x^3 - 5x^2 + 4x}{-2} \right) \\
 &\quad + 5 \frac{x^3 - 3x^2 + 2x}{24} \\
 &= \frac{1}{12} (x^3 - 9x^2 + 8x - 12)
 \end{aligned}$$

Numerical Quadrature

In general, a numerical integration is the approximation of a definite integration by a weighted sum of function values at discretized within the interval of integration.

$$\int_a^b f(x) dx = \sum_{i=0}^n w_i f(x_i)$$

Where w_i is the weighted factor depending on the integration schemes used and $f(x_i)$ is the function values evaluated at the given point x_i .

The term numerical quadrature is synonym for numerical integration, especially as applied to one dimensional integration.

Q.24. If $f(x)$ is a polynomial of degree 2, prove that

$$\int_0^1 f(x) dx = \frac{1}{12} [5f(0) + 8f(1) - f(2)]$$

Sol : We know that

$$\int_0^1 f(x) dx = \sum_{i=0}^2 A_i f(x_i) \quad \dots (*)$$

for polynomial of degree 2

Here $x_0 = 0, x_1 = 1, x_2 = 2$

& $f(x) = x^k$ for $k = 0, 1, 2$

Now

$$I_0 = \int_0^1 x^0 dx = [x]_0^1 = 1$$

$$I_1 = \int_0^1 x dx = \frac{1}{2} [x^2]_0^1 = \frac{1}{2}$$

$$I_2 = \int_0^1 x^2 dx = \frac{1}{3} [x^3]_0^1 = \frac{1}{3}$$

Again

$$\begin{aligned}
 I_0 &= \sum_{i=0}^2 A_i x_i^0 = \sum_{i=0}^2 A_i x_i^0 \quad \text{By } (*) \\
 1 &= A_0 + A_1 + A_2 \quad \dots (1) \quad \because I_0 = 1
 \end{aligned}$$

&

$$\begin{aligned}
 I_1 &= \sum_{i=0}^2 A_i x_i = A_0 x_0 + A_1 x_1 + A_2 x_2 \\
 \frac{1}{2} &= A_0 \cdot 0 + A_1 \cdot 1 + A_2 \cdot 2 \quad \because I_1 = \frac{1}{2}
 \end{aligned}$$

$$\frac{1}{2} = A_1 + 2A_2 \Rightarrow 1 = 2A_1 + 4A_2 \quad \dots (2)$$

&

$$\begin{aligned}
 I_2 &= \sum_{i=0}^2 A_i x_i^2 = A_0 x_0^2 + A_1 x_1^2 + A_2 x_2^2 \\
 \frac{1}{3} &= A_0 (0)^2 + A_1 (1)^2 + A_2 (2)^2 \\
 \frac{1}{3} &= A_1 + 4A_2 \Rightarrow 1 = 3A_1 + 12A_2 \quad \dots (3)
 \end{aligned}$$

By (3) - (2) $\times$ (2)

$$\begin{array}{r}
 3A_1 + 12A_2 = 1 \\
 + 6A_1 + 12A_2 = +3 \\
 \hline
 -3A_1 = -2 \Rightarrow A_1 = \frac{2}{3}
 \end{array}$$

$$\text{Put in (2)} \quad 1 = 2 \left(\frac{2}{3} \right) + 4A_2 \Rightarrow A_2 = -\frac{1}{12}$$

$$\text{Put } A_1 = \frac{2}{3} \text{ and } A_2 = -\frac{1}{12} \text{ in (1)}$$

$$1 = A_0 + \frac{2}{3} - \frac{1}{12} \Rightarrow A_0 = \frac{5}{12}$$

By (*)

$$\int_0^1 f(x) dx = A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2)$$

$$= \frac{5}{12} f(0) + \frac{2}{3} f(1) - \frac{1}{12} f(2)$$

$$\Rightarrow \int_0^1 f(x) dx = \frac{1}{12} [5f(0) + 8f(1) - f(2)]$$

Quadrature Formula (Newton's Cote's Quadrature Formula)

Given that

$$I = \int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_n} y dx$$

with n -equal intervals and $x_n = x_0 + nh$

(1) Trapezoidal Rule :

$$I = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

(2) Simpson's One Third Rule :

$$I = \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + y_5 + \dots)]$$

(3) Simpson's Three-eight Rule :

$$I = \frac{3h}{8} [y_0 + y_n + 2(y_3 + y_6 + y_9 + \dots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \dots)]$$

(4) Weddle Rule : n is multiple of 6 :

(A) When $n = 6$

$$I = \frac{3h}{10} [(y_0 + y_6) + 5(y_1 + y_5) + y_2 + y_4 + 2y_3]$$

(B) $n = 12$

$$I = \frac{3h}{10} [(y_0 + y_{12}) + 5(y_1 + y_9 + y_5 + y_{11}) + (y_2 + y_4 + y_8 + y_{10}) + 2y_6 + 6(y_3 + y_7)]$$

Q. 25. Prove that the sum of cote's number is unity.

OR

Prove that $\sum_{k=0}^n C_k^n = 1$

Sol : We know that

$$C_k^n = \frac{1}{n} \int_0^n L_k du \quad \dots(1)$$

Therefore

$$\sum_{k=0}^n C_k^n = C_0^n + C_1^n + C_2^n + \dots + C_n^n$$

$$\sum_{k=0}^n C_k^n = \frac{1}{n} \left[\int_0^n [L_0 + L_1 + L_2 + \dots + L_n] du \right] \quad \dots(*)$$

Also we know that

$$f(x) = \sum_{k=0}^n L_k(x) Y_k \quad \dots(2)$$

Where

$$x_k = x_0 + kh$$

& Let

$$f(x) = (x - \alpha)^i$$

then

$$y_k = (x_k - \alpha)^i$$

Put in (2)

$$(x - \alpha)^i = \sum_{k=0}^n L_k(x) [x_k - \alpha]^i$$

Put $i = 0$

$$(x - \alpha)^0 = \sum_{k=0}^n L_k(x) [x_k - \alpha]^0$$

$$I = \sum_{k=0}^n L_k(x)$$

$$I = L_0 + L_1 + L_2 + \dots + L_n$$

Or

Put in (*)

$$= \sum_{k=0}^n C_k^n = \frac{1}{n} \left[\int_0^n [1] du \right]$$

$$= \frac{1}{n} [u]_0^n = 1$$

$$\Rightarrow \sum_{k=0}^n C_k^n = 1$$

Q.26. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using (1) Trapezoidal rule

(2) Simpson rule (both) (3) Weddle rule, where the interval of integration is subdivided into 6 equal parts.

Sol : Here $\int_a^b f(x) dx = \int_{x_0}^{x_n} f(x) dx = \int_0^1 \frac{dx}{1+x^2}$

i.e. $x_0 = 0, x_n = x_6 = 1$ and by $x_n = x_0 + nh$

$$\Rightarrow h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$$\& y = f(x) = \frac{1}{1+x^2}$$

$$x_0 = 0 \Rightarrow y_0 = \frac{1}{1+0} = 1$$

$$\text{and } x_1 = \frac{1}{6} \Rightarrow y_1 = \frac{1}{1+\left(\frac{1}{6}\right)^2} = \frac{36}{37} = 0.97297$$

$$x_2 = \frac{2}{6} = \frac{1}{3} \Rightarrow y_2 = \frac{1}{1+\left(\frac{1}{3}\right)^2} = \frac{9}{10} = 0.9$$

$$x_3 = \frac{3}{6} = \frac{1}{2} \Rightarrow y_3 = \frac{1}{1+\left(\frac{1}{2}\right)^2} = \frac{4}{5} = 0.8$$

$$x_4 = \frac{4}{6} = \frac{2}{3} \Rightarrow y_4 = \frac{1}{1+\left(\frac{2}{3}\right)^2} = \frac{9}{15} = 0.6923$$

$$x_5 = \frac{5}{6} \Rightarrow y_5 = \frac{1}{1+\left(\frac{5}{6}\right)^2} = \frac{36}{61} = 0.59016$$

$$\& \quad x_6 = 1 \Rightarrow y_6 = \frac{1}{1+1^2} = 0.5$$

(1) Trapezoidal Rule :

$$I = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1/6}{2} [(1 + 0.5) + 2(0.97297 + 0.9 + 0.8 + 0.6923 + 0.59016)]$$

$$\Rightarrow I = 0.78424$$

(2) Simpson's 1/3 Rule :

$$I = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1/6}{3} [(1 + 0.5) + 2(0.9 + 0.6923) + 4(0.97297 + 0.8 + 0.59016)]$$

$$I = 0.78539$$

(3) Simpson's Rule :

$$I = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3(1/6)}{8} [(1 + 0.5) + 2(0.8) + 3(0.97297 + 0.9 + 0.6923 + 0.59016)]$$

$$= 0.7853593$$

(4) Weddle's Rule :

$$I = \frac{3h}{10} [(y_0 + y_6) + 5(y_1 + y_5) + y_2 + y_4 + 6y_3]$$

$$I = \frac{3(1/6)}{10} [(1 + 0.5) + 5(0.97297 + 0.59016) + 0.9 + 0.6923 + 6(0.8)]$$

$$= I = 0.7853975$$

Q.28. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (1) Trapezoidal rule (2) Simpson's 1/3 rule (3) Simpson's 2/3 rule (4) Weddle rule

Sol. : Given that : $I = \int_0^6 \frac{dx}{1+x^2} = \int_{x_0}^{x_n} f(x) dx$

$$\Rightarrow \quad f(x) = y = \frac{1}{1+x^2} \quad \text{---(1)}$$

& $x_0 = 0, x_n = 6$ Let $n = 6$ then for $x_n = x_0 + nh$

$$h = \frac{x_n - x_0}{n} = \frac{6-0}{6} = 1.0$$

$$x_0 = 0 \Rightarrow y_0 = \frac{1}{1+0} = 1.00$$

$$\& \quad x_1 = x_0 + 1.h \Rightarrow x_1 = 1 \Rightarrow y_1 = \frac{1}{1+1^2} = 0.5$$

$$x_2 = x_0 + 2h \Rightarrow x_2 = 2 \Rightarrow y_2 = \frac{1}{1+2^2} = 0.2$$

$$x_3 = x_0 + 3h \Rightarrow x_3 = 3 \Rightarrow y_3 = \frac{1}{1+3^2} = 0.1$$

$$x_4 = x_0 + 4h \Rightarrow x_4 = 4 \Rightarrow y_4 = \frac{1}{1+4^2} = 0.058824$$

$$x_5 = x_0 + 5h \Rightarrow x_5 = 5 \Rightarrow y_5 = \frac{1}{1+5^2} = 0.038462$$

$$x_6 = 6 \Rightarrow y_6 = \frac{1}{1+6^2} = 0.027027$$

(1) Trapezoidal Rule

$$I = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$I = \frac{1}{2} [(1 + 0.027027) + 2(0.5 + 0.2 + 0.1 + 0.058824 + 0.038462)]$$

$$I = 1.4107995$$

(2) Simpson's 1/3 Rule

$$I = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$I = \frac{1}{3} [(1 + 0.0270270) + 2(0.2 + 0.058824) + 4(0.5 + 0.1 + 0.038462)]$$

$$I = 1.3661743$$

(3) Simpson's 3/8 Rule :

$$I = \frac{3h}{8} [(1y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$I = \frac{3(1)}{8} [(1 \cdot 0.027027) + 2(0.1) + 3(0.5 + 0.2 + 0.058824 + 0.038462)]$$

$$I = 1.3570819$$

(4) Weddle's Rule

$$I = \frac{3h}{10} [(y_0 + y_6) + 2y_3 + 5(y_1 + y_5) + y_2 + y_4]$$

$$= \frac{3(1)}{10} [(1 + 0.027027) + 2(0.1) + 5(0.5 + 0.038462) + 0.2 + 0.058824]$$

$$\Rightarrow I = 1.3734483$$

Q.29. Evaluate $\int_{-3}^3 x^4 dx$ by using (1) Trapezoidal (2)

Simpson's rule :

Sol. : Given that

$$I = \int_{-3}^3 x^4 dx = \int_{x_0}^{x_n} f(x) dx$$

$$\text{where } f(x) = x^4, x_0 = -3, x_n = 3$$

$$\therefore x_n = x_0 + nh, \text{ Let } n = 6,$$

$$\Rightarrow h = \frac{x_n - x_0}{n} = \frac{3 - (-3)}{6} = 1$$

$$\therefore x_0 = -3 \Rightarrow y_0 = (-3)^4 = 81$$

$$x_1 = x_0 + 1 \cdot h \Rightarrow x_1 = -3 + 1 = -2 \Rightarrow y_1 = (-2)^4 = 16$$

$$x_2 = x_0 + 2h \Rightarrow x_2 = -3 + 2 = -1 \Rightarrow y_2 = (-1)^4 = 1$$

$$x_3 = 0 \Rightarrow y_3 = (0)^4 = 0$$

$$x_4 = 1 \Rightarrow y_4 = (1)^4 = 1$$

$$x_5 = 2 \Rightarrow y_5 = (2)^4 = 16$$

$$x_6 = 3 \Rightarrow y_6 = (3)^4 = 81$$

&
then by

(1) Trapezoidal rule :

$$I = \frac{h}{2} [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$I = \frac{1}{2} [(81 + 81) + 2(16 + 1 + 0 + 1 + 16)]$$

$$I = 98$$

(2) Simpson's 1/3 Rule :

$$I = \frac{h}{3} [(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{3} [(81 + 81) + 2(1 + 1) + 4(16 + 0 + 16)]$$

$$I = 98$$

(3) Simpson's 3/8 Rule :

$$I = \frac{3h}{8} [(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_3 + y_5)]$$

$$= \frac{3}{8} [(81 + 81) + 2(0) + 3(16 + 1 + 1 + 16)]$$

$$I = 99$$

Actual Value :

$$I = \int_{-3}^3 x^4 dx = \left[\frac{x^5}{5} \right]_{-3}^3$$

$$= \frac{1}{5} [(3)^5 - (-3)^5] = \frac{1}{5} [486]$$

$$I = 97.2$$

Q. 30. Calculate $\int_0^1 \frac{dx}{1+x}$ by Simpson's 1/3 Rule taking 10 equal parts.

Sol. : Here $f(x) = \frac{1}{1+x}$, $x_0 = 0$, $x_n = 1$, $n = 10$

$$h = \frac{x_0 - x_9}{10} = \frac{1 - 0}{10} = 0.1$$

$$x_n = x_0 + nh$$

$$x_0 = 0 \Rightarrow y_0 = \frac{1}{1+0} = 1$$

$$x_1 = 0.1 \Rightarrow y_1 = \frac{1}{1+0.1} = 0.909090$$

$$x_2 = 0.2 \Rightarrow y_2 = \frac{1}{1+0.2} = 0.833333$$

$$x_3 = 0.3 \Rightarrow y_3 = \frac{1}{1+0.3} = 0.76923$$

$$x_4 = 0.4 \Rightarrow y_4 = \frac{1}{1+0.4} = 0.71428$$

$$x_5 = 0.5 \Rightarrow y_5 = \frac{1}{1+0.5} = 0.66666$$

$$x_6 = 0.6 \Rightarrow y_6 = \frac{1}{1+0.6} = 0.625$$

$$x_7 = 0.7 \Rightarrow y_7 = \frac{1}{1+0.7} = 0.57143$$

$$x_8 = 0.8 \Rightarrow y_8 = \frac{1}{1+0.8} = 0.55555$$

$$x_9 = 0.9 \Rightarrow y_9 = \frac{1}{1+0.9} = 0.52632$$

$$x_{10} = 1 \Rightarrow y_{10} = \frac{1}{1+1} = 0.5$$

Now by Simpson's 1/3 Rule

$$I = \frac{h}{3} [(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8)] + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$I = \frac{(0.1)}{3} [(1 + 0.5) + 2(0.83333 + 0.71428 + 0.625 + 0.55555) + 4(0.90909 + 0.76923 + 0.66666 + 0.57143 + 0.52632)]$$

$$I = 0.69315$$

Ans. ■

Objectives

The value of $\Delta f(x)$ is :

1. (a) $f(x+h) - f(x)$ (b) $f(x) - f(x+h)$
(c) $f(x-h) - f(x)$ (d) $f(x-h) + f(x)$

The central difference operator δ equals :

2. (a) $E^{1/2} + E^{-1/2}$ (b) $\frac{E^{1/2} - E^{-1/2}}{2}$
(c) $E^{1/2} - E^{-1/2}$ (d) $\frac{E^{1/2} + E^{-1/2}}{2}$

The averaging operator μ equals :

3. (a) $E^{1/2} + E^{-1/2}$ (b) $\frac{E^{1/2} - E^{-1/2}}{2}$
(c) $E^{1/2} - E^{-1/2}$ (d) $\frac{E^{1/2} + E^{-1/2}}{2}$

The value of backward difference operator $\nabla f(x)$ is :

4. (a) $f(x-h) - f(x)$ (b) $f(x+h) - f(x)$
(c) $f(x) - f(x-h)$ (d) $f(x) - f(x+h)$

The relation among E , Δ and ∇ is :

5. (a) $E\Delta = \Delta E = \nabla + E$ (b) $\Delta E = E\Delta = \nabla - E$
(c) $E\nabla = \nabla E = \Delta$ (d) $\Delta\nabla = E + 1$

The relation between Δ and ∇ is :

6. (a) $\Delta\nabla = \Delta - \nabla$ (b) $(1 + \Delta)(1 - \nabla) = 1$
(c) (a) and (b) (d) None of these

In the usual notations, choose the correct choice of E :

7. (a) $E + 1 + \Delta$ (b) $E = 1 - \Delta$
(c) $E = 1 + \nabla$ (d) $E = 1 - \nabla$

If $y = p(x)$ is a polynomial of degree n then its $(n+1)$ th divided difference is :

8. (a) A constant (b) Zero
(c) Function of x (d) None of these

The sum of lagrangian coefficient is :

9. (a) $n\beta$ (b) 1
(c) -1 (d) None of these

The forward difference operator Δ is :

10. (a) $\mu\delta + \frac{\delta^2}{2}$ (b) $\mu\delta - \frac{\delta^2}{2}$

- (c) $\mu\delta + \delta^2$ (d) None of these
11. The n^{th} divided difference of x^n is :
 (a) $n!$ (b) n
 (c) β (d) None
12. The differentiating interval h in calculus of finite difference is :
 (a) Finite and small (b) Finite and large
 (c) Tends to zero (d) None
13. Newton's forward interpolation formula gives better results :
 (a) Near the beginning of a set of tabular values
 (b) Near the end of a set of tabular values
 (c) Near the middle of a table
 (d) None of these
14. Newton's backward difference interpolation formula gives better results :
 (a) Near the beginning of a set of tabular values
 (b) Near the end of a set of tabular values
 (c) Near the middle of a table
 (d) None of these
15. Value of $\Delta \log x$ is :
 (a) $\log \left(1 - \frac{h}{x}\right)$ (b) $\log \left(1 + \frac{h}{x}\right)$
 (c) $\log \frac{h}{x}$ (d) $\log \frac{x}{h}$
16. The value of $\Delta \tan^{-1} x$ is :
 (a) $\frac{1}{1+x^2}$ (b) $-\frac{1}{1+x^2}$
 (c) $\tan^{-1} \left(\frac{h}{1+x^2}\right)$ (d) $\tan^{-1} \left(\frac{h}{1+hx+x^2}\right)$
17. The third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$ is :
 (a) 1 (b) -1
 (c) 0 (d) 8
18. Using Newton's divided difference interpolation the required polynomial corresponding to $f(0) = 1$, $f(1) = 3$, $f(3) = 55$ is :
 (a) $8x^2 - 6x + 1$ (b) $x^2 - 6x + 1$
 (c) $8x^2 + 6x + 1$ (d) None

19. The n^{th} divided difference of $f(x) = \frac{1}{x}$ is :
 (a) $\frac{(-1)^n}{x_0 x_1 x_2 \dots x_n}$ (b) $\frac{1}{x_0 x_1 x_2 \dots x_n}$
 (c) $\frac{(-1)^n}{x_0 x_1 x_2 \dots x_{n-1}}$ (d) $\frac{1}{x_0 x_1 x_2 \dots x_{n-1}}$
20. ∇^2 is equal to :
 (a) $(1 - E^{-1})^2$ (b) $(E + 1)^2$
 (c) $(E - 1)^2$ (d) None of these
21. Δ^2 is equal to :
 (a) $(E - 1)^3$ (b) $(E + 1)^2$
 (c) $(E - 1)^2$ (d) None of these
22. The value of $\Delta_{yz} x^2$ is :
 (a) $x^2 + y^2 + z^2$ (b) $x + y + z$
 (c) 0 (d) 1
23. Simpson's one third formula is also called :
 (a) Elliptic Formula (b) Parabolic formula
 (c) Hyperbolic formula (d) Cubic formula
24. The sum of cote's number is :
 (a) 1 (b) n (c) 2^n (d) 0
25. The value of $\int_0^1 \frac{dx}{1+x^2}$ by simpson's 1/3 rule ($h = 1/4$) is :
 (a) 0.8702 (b) 0.7853 (c) 0.7954 (d) None of these
26. Using simpson's 1/3 rule, the interval h for better approximation should be :
 (a) Even and small (b) Odd and small
 (c) None (d) Even and large

Answer

- | | | | | |
|--------|--------|--------|--------|--------|
| 1.(a) | 2.(c) | 3.(d) | 4.(c) | 5.(c) |
| 6.(c) | 7.(a) | 8.(b) | 9.(b) | 10.(a) |
| 11.(c) | 12.(a) | 13.(a) | 14.(b) | 15.(b) |
| 16.(d) | 17.(a) | 18.(a) | 19.(a) | 20.(a) |
| 21.(b) | 22.(d) | 23.(b) | 24.(a) | 25.(b) |
| 26.(a) | | | | |

□□□

Unit-III

Methods to Solve of Linear Equations: 1. Direct methods for solving system of linear equations: (i) Gauss elimination, (ii) LU decomposition, (iii) Cholesky decomposition. 2. Iterative methods: (i) Jacobi, (ii) Gauss-Seidal.

A system of n-linear algebraic equations in n-variable is given below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= d_2 \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &= d_3 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= d_n \end{aligned}$$

Solution of linear simultaneous equations: [Direct Method]

(1) Gauss-Elimination Method:

$$\text{Let } \begin{cases} ax + by + cz = d \\ a^1x + b^1y + c^1z = d^1 \\ a_1x + b_1y + c_1z = d_1 \end{cases}$$

be a system of linear equation of three variables

Its matrix form

$$\begin{bmatrix} a & b & c \\ a^1 & b^1 & c^1 \\ a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ d^1 \\ d_1 \end{bmatrix} \quad \dots(*)$$

$$\text{where } A = \begin{bmatrix} a & b & c \\ a^1 & b^1 & c^1 \\ a_1 & b_1 & c_1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and } C = \begin{bmatrix} d \\ d^1 \\ d_1 \end{bmatrix}$$

$$\text{Let } C = [A : B]$$

$$C = \begin{bmatrix} a & b & c & d \\ a^1 & b^1 & c^1 & d^1 \\ a_1 & b_1 & c_1 & d_1 \end{bmatrix}$$

By elementary transformation in Row, reduce C into following form

$$C \sim \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 0 & f_1 & f_2 & f_3 \\ 0 & 0 & g_1 & g_2 \end{bmatrix}$$

and Put in (*) & find the values of x, y, z.

(2) Gauss-Jordan Method:

As above taking

$$C \sim \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 0 & f_1 & f_2 & f_3 \\ 0 & 0 & g_1 & g_2 \end{bmatrix}$$

Reduce C into following form

$$C \sim \begin{bmatrix} e & 0 & 0 & e^1 \\ 0 & f & 0 & f^1 \\ 0 & 0 & g & g^1 \end{bmatrix}$$

and put in (*) & find the values of x, y, z.

Q.1. Solve the following system of equations by Gauss elimination method:

$$(1) \quad \begin{cases} 2x - y + 3z = 90 \\ x + y + z = 6 \\ x - y + z = 2 \end{cases}$$

$$(2) \quad \begin{cases} 10x + y + 2z = 13 \\ 3x + 10y + z = 14 \\ 2x + 3y + 10z = 15 \end{cases}$$

$$(3) \quad \begin{cases} 3x + 6y + z = 16 \\ 2x + 4y + 3z = 13 \\ x + 3y + 2z = 9 \end{cases}$$

$$\text{Sol.: (1)} \quad \begin{cases} 2x - y + 3z = 9 \\ x + y + z = 6 \\ x - y + z = 2 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} \quad \dots(*)$$

$$AX = C \quad \dots(1)$$

when

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad C = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

Let

$$B = [A : C]$$

$$B = \begin{bmatrix} 2 & -1 & 3 & 9 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

$$R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

$$R_2 - R_1, R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 3 & -1 & 3 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

$$R_3 - \frac{1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 2 & 3 \\ 0 & 3 & -1 & 3 \\ 0 & 1 & -2/3 & -2 \end{bmatrix}$$

$$\text{Now, by (*)} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix}$$

$$x - 2y + 2z = 3 \quad \dots(1)$$

$$3y - z = 3 \quad \dots(2)$$

$$-\frac{2}{3}z = -2 \Rightarrow z = 3$$

$$\text{Put } z = 3 \text{ in (2)}$$

$$3y - 3 = 3 \Rightarrow y = 2$$

$$\text{Put } y = 2 \text{ and } z = 3 \text{ in (1)}$$

$$x - 2(2) + 2(3) = 3 \Rightarrow x = 1$$

$$\text{Ans : } x = 1, y = 2, z = 3$$

$$(2) \begin{cases} 10x + y + 2z = 13 \\ 3x + 10y + z = 14 \\ 2x + 3y + 10z = 15 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 10 & 1 & 2 \\ 3 & 10 & 1 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \\ 15 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\begin{bmatrix} 1 & -29 & -1 \\ 3 & 10 & 1 \\ 2 & 3 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

$$R_2 - 3R_1, R_3 - 2R_1$$

$$\begin{bmatrix} 1 & -29 & -1 \\ 0 & 97 & 4 \\ 0 & 61 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -29 \\ 101 \\ 73 \end{bmatrix}$$

$$R_3 - \frac{61}{97} R_2$$

$$\begin{bmatrix} 1 & -29 & -1 \\ 0 & 97 & 4 \\ 0 & 0 & 920 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -29 \\ 101 \\ 920 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x - 29y - z = -29 & \dots(1) \\ 97y + 4z = 101 & \dots(2) \\ 920z = 920 \Rightarrow z = 1 & \dots(3) \end{cases}$$

$$\text{Put } z = 1 \text{ in (2)} \quad 97y + 4(1) = 101 \Rightarrow y = 1$$

$$\text{Put } y = 1, z = 1 \text{ in (1)} \quad x - 29(1) - 1 = -29 \Rightarrow x = 1$$

$$\text{Ans. } x = 1, y = 1, z = 1$$

$$Q.2. \begin{cases} 3x + 6y + z = 16 \\ 2x + 4y + 3z = 13 \\ x + 3y + 2z = 9 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 3 & 6 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \\ 9 \end{bmatrix}$$

$$R_1 - R_2$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 4 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 9 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 6 \end{bmatrix}$$

$$x + 2y - 2z = 3 \quad \dots(1)$$

$$\Rightarrow 7z = 7 \quad \dots(2)$$

$$y + 4z = 6 \quad \dots(3)$$

$$\text{By (2), } z = 1, \text{ put in (3)}$$

$$y + 4(1) = 6 \Rightarrow y = 2$$

Put $y = 2, z = 1$ in (1) $x + 2(2) - 2(1) = 3$

$$x = 2$$

Ans. $x = 1, y = 2, z = 1$

Q.3. Solve the following system of equations by Gauss-Jordan Method :

$$(1) \begin{cases} x + 2y + z = 8 \\ 2x + 3y + 4z = 20 \\ 4x + 3y + 2z = 16 \end{cases}$$

$$(2) \begin{cases} 10x + y + z = 12 \\ 2x + 10y + z = 13 \\ x + y + 5z = 7 \end{cases}$$

$$(3) \begin{cases} x + y + z = 9 \\ 2x - 3y + 4z = 13 \\ 3x + 4y + 5z = 40 \end{cases}$$

$$\text{sol. : (1)} \begin{cases} x + 2y + z = 8 \\ 2x + 3y + 4z = 20 \\ 4x + 3y + 2z = 16 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 16 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -16 \end{bmatrix}$$

$$R_1 + 2R_2, R_3 - 5R_2$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ -36 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{-12} R_3$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 16 \\ 04 \\ 03 \end{bmatrix}$$

$$R_1 - 5R_3, R_2 - 2R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 1, y = 2, z = 3$$

Ans.

$$(2) \begin{cases} 10x + y + z = 12 \\ 2x + 10y + z = 13 \\ x + y + 5z = 7 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 2 & 10 & 1 \\ 10 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \\ 12 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 10R_1$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 8 & -9 \\ 0 & -9 & -49 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -58 \end{bmatrix}$$

$$R_2 + R_3$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & -1 & -58 \\ 0 & -9 & -49 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -59 \\ -58 \end{bmatrix}$$

$$R_2 \rightarrow (-1) R_2, R_3 \rightarrow (-1) R_3$$

$$\begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 58 \\ 0 & 9 & 49 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 59 \\ 58 \end{bmatrix}$$

$$R_1 - R_2, R_3 - 9R_2$$

$$\begin{bmatrix} 1 & 0 & -53 \\ 0 & 1 & 58 \\ 0 & 0 & -473 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -52 \\ 59 \\ -473 \end{bmatrix}$$

$$R_1 \rightarrow (-1)R_1, R_3 \rightarrow \left(-\frac{1}{473}\right) R_3$$

$$\begin{bmatrix} -1 & 0 & 53 \\ 0 & 1 & 58 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 52 \\ 59 \\ 1 \end{bmatrix}$$

$$R_1 - 53R_3, R_2 - 58R_3$$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$$

$$(3) \quad \begin{cases} x + y + z = 9 \\ 2x - 3y + 4z = 13 \\ 3x + 4y + 5z = 40 \end{cases}$$

Its matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$R_2 - 2R_1, R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ -5 \end{bmatrix}$$

$$R_1 - R_2, R_3 + 5R_2$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 13 \\ 60 \end{bmatrix}$$

$$R_3 \rightarrow \frac{1}{12} R_3$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 13 \\ 5 \end{bmatrix}$$

$$R_1 + R_3, R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x = 1 \\ y = 3 \\ z = 5 \end{cases}$$

Ans.

Ans.

LU - Decomposition Method [Factorization Method]

Let

$$\begin{cases} ax + by + cz = d \\ a^1x + b^1y + c^1z = d^1 \\ a_1x + b_1y + c_1z = d_1 \end{cases}$$

be a system of linear equation

Its matrix form

$$\begin{bmatrix} a & b & c \\ a^1 & b^1 & c^1 \\ a_1 & b_1 & c_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ d^1 \\ d_1 \end{bmatrix}$$

$$AX = B$$

...(1)

$$\text{where } A = \begin{bmatrix} a & b & c \\ a^1 & b^1 & c^1 \\ a_1 & b_1 & c_1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d \\ d^1 \\ d_1 \end{bmatrix}$$

Put $A = LU$

...(2)

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

&

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

By (1) & (2)

$$\begin{aligned} \text{Put } LUX &= B \quad \dots(3) \\ UX &= Y \quad \dots(4) \end{aligned}$$

$$\text{where } Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

By (3) & (4)

$$LY = B$$

Find the values of x, y, z from (4)

3. Solve the following system of equations by LU decomposition method $x + 5y + z = 14$, $2x + y + 3z = 13$, $3x + y + 4z = 17$.

Sol. : Given that

$$\begin{aligned} x + 5y + z &= 14 \\ 2x + y + 3z &= 13 \\ 3x + y + 4z &= 17 \end{aligned}$$

Its matrix form

$$\begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

Put

$$A = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

&

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$A = LU$$

$$\begin{aligned} \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} &= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \end{aligned}$$

On comparison

$$\begin{aligned} u_{11} &= 1, u_{12} = 5, u_{13} = 1 \\ l_{21}u_{11} &= 2 \Rightarrow l_{21}(1) = 2 \Rightarrow l_{21} = 2 \\ l_{31}u_{11} &= 3 \Rightarrow l_{31}(1) = 3 \Rightarrow l_{31} = 3 \\ l_{21}u_{12} + u_{22} &= 1 \Rightarrow 2(5) + u_{22} = 1 \Rightarrow u_{22} = -9 \\ l_{31}u_{12} + l_{32}u_{22} &= 1 \Rightarrow 3(5) + l_{32}(-9) = 1 \Rightarrow l_{32} = 14/9 \\ l_{21}u_{13} + u_{23} &= 3 \Rightarrow 2(1) + u_{23} = 3 \Rightarrow u_{23} = 1 \end{aligned}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4 \Rightarrow 3(1) + \left(\frac{14}{9}\right)1 + u_{33} = 4 \Rightarrow u_{33} = -5/9$$

By (1) & (2)

$$\begin{aligned} LUX &= B \quad \dots(3) \\ UX &= Y \quad \dots(4) \end{aligned}$$

Put

$$\text{where } Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

By (3) & (4)

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 17 \end{bmatrix}$$

 $\Rightarrow$

$$\begin{aligned} u &= 14 \\ 2u + v &= 13 \Rightarrow v = -15 \end{aligned}$$

$$\text{and } 3u + \frac{14}{9}v + w = 17$$

$$\Rightarrow w = -5/3$$

Now by (4)

$$UX = Y$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -5/9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ -15 \\ -5/3 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x + 5y + z &= 14 \\ -9y + z &= -15 \\ -\frac{5}{9}z &= -\frac{5}{3} \Rightarrow z = 3 \end{aligned}$$

Put in (6)

$$-9y + 3 = -15 \Rightarrow y = 2$$

Put $y = 2$ and $z = 3$ in (5) $x + 5(2) + 3 = 14$

$$\Rightarrow x = 1$$

Ans. $x = 1, y = 2, z = 3$

Q.4. Solve the following system by triangularization method :

$$x + y + z = 1, 4x + 3y - z = 6, 3x + 5y + 3z = 4$$

Sol. : Its matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

Put

where

$$A = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

By (2)

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

On comparison

$$u_{11} = 1, u_{12} = 1, u_{13} = 1$$

$$l_{21}u_{11} = 4 \Rightarrow l_{21}(1) = 4 \Rightarrow l_{21} = 4$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31}(1) = 3 \Rightarrow l_{31} = 3$$

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow 4(1) + u_{22} = 3 \Rightarrow u_{22} = -1$$

$$l_{31}u_{12} + l_{32}u_{22} = 5 \Rightarrow 3(1) + l_{32}(-1) = 5 \Rightarrow l_{32} = -2$$

$$l_{21}u_{13} + u_{23} = -1 \Rightarrow 4(1) + u_{23} = -1 \Rightarrow u_{23} = -5$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow 3(1) + (-2)(-5) + u_{33} = 3 \Rightarrow u_{33} = -10$$

$$\text{By (1) \& (2)} \quad LUX = B \quad \dots(3)$$

Put

$$UX = Y \quad \dots(4)$$

$$\text{where } Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

By (3) \& (4) $LY = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

 $\Rightarrow$

$$u = 1$$

$$4u + v = 6 \Rightarrow v = 2$$

$$\text{and } 3u - 2v + w = 4 \Rightarrow w = 5$$

Now by (4)

$$UX = Y$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

 $\Rightarrow$

$$x + y + z = 1$$

$$-y + 5z = 2$$

$$-10z = 5 \Rightarrow z = -1/2$$

$$\dots(5)$$

$$\dots(6)$$

$$\text{Put } z = -1/2 \text{ in (6) } -y + 5\left(-\frac{1}{2}\right) = 2 \Rightarrow y = \frac{1}{2}$$

$$\text{Put } y = \frac{1}{2}, z = -\frac{1}{2} \text{ in (5) } x = 1$$

$$\text{Ans. } x = 1, y = \frac{1}{2}, z = -\frac{1}{2}$$

Q.5. Apply LU decomposition method to solve the equation.

$$3x_1 - 2x_2 + x_3 = 10, 2x_1 + 3x_2 + 2x_3 = 14, x_1 + 2x_2 + 3x_3 = 14$$

Sol. : Its matrix form

$$\begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$\text{Put } A = LU$$

$$\text{or } \begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 3 & -2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

On Comparison

$$u_{11} = 3, u_{12} = 2, u_{13} = 1$$

$$l_{21}u_{11} = 2 \Rightarrow l_{21}(3) = 2 \Rightarrow l_{21} = 2/3$$

$$l_{31}u_{11} = 1 \Rightarrow l_{31}(3) = 1 \Rightarrow l_{31} = 1/3$$

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow \frac{2}{3}(2) + u_{22} = 3 \Rightarrow u_{22} = \frac{5}{3}$$

$$l_{31}u_{12} + l_{32}u_{22} = 2 \Rightarrow \left(\frac{1}{3}\right)(2) + l_{32}\left(\frac{5}{3}\right) = 2 \Rightarrow l_{32} = \frac{4}{5}$$

$$l_{21}u_{13} + u_{23} = 2 \Rightarrow \frac{2}{3}(1) + u_{23} = 2 \Rightarrow u_{23} = \frac{4}{3}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow \left(\frac{1}{3}\right)(1) + \frac{4}{5}\left(\frac{4}{3}\right) + u_{33} = 3$$

$$\Rightarrow u_{33} = \frac{24}{15}$$

By (1) & (2)

$$LUX = B \quad \dots(3)$$

Put

$$UX = Y \quad \dots(4)$$

By (3) & (4)

$$LY = B \quad \text{where } Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$u = 10$$

$$\Rightarrow 2u + v = 14 \Rightarrow v = \frac{22}{3}$$

$$\text{and } \frac{1}{3}u + \frac{4}{5}v + w = 14 \Rightarrow w = 72/15$$

Now by (4)

$$UX = Y$$

$$\begin{bmatrix} 5 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 24/15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 72/15 \end{bmatrix}$$

$$\Rightarrow 3x_1 + 2x_2 + x_3 = 10 \quad \dots(5)$$

$$\frac{5}{3}x_2 + \frac{4}{3}x_3 = \frac{22}{3} \text{ or } 5x_2 + 4x_3 = 22 \quad \dots(6)$$

$$\frac{24}{15}x_3 = \frac{72}{15} \Rightarrow x_3 = 3$$

$$\text{Put in (6) } 5x_2 + 4(3) = 22 \Rightarrow x_2 = 2$$

$$\text{Put } x_2 = 2, x_3 = 3 \text{ in (5) } 3x_1 + 2(2) + 3 = 10 \Rightarrow x_1 = 1$$

$$\text{Ans. } x_1 = 1, x_2 = 2, x_3 = 3$$

Q.6. Apply LU decomposition method to solve the equation :

$$3x + 2y + 7z = 4, 2x + 3y + z = 5, 3x + 4y + z = 7$$

Sol. : Its matrix form

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$AX = B$$

...(1)

Where $A = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Put

$$A = LU$$

...(2)

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

By (2)

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

On Comparison

$$u_{11} = 3, u_{12} = 2, u_{13} = 7$$

$$l_{21}u_{11} = 2 \Rightarrow l_{21}(3) = 2 \Rightarrow l_{21} = 2/3$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31}(3) = 3 \Rightarrow l_{31} = 1$$

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow \frac{2}{3}(2) + u_{22} = 3 \Rightarrow u_{22} = \frac{5}{3}$$

$$l_{31}u_{12} + l_{32}u_{22} = 4 \Rightarrow 1(2) + l_{32}\left(\frac{5}{3}\right) = 4 \Rightarrow l_{32} = 6/5$$

$$l_{21}u_{13} + u_{23} = 1 \Rightarrow \frac{2}{3}(7) + u_{23} = 1 \Rightarrow u_{23} = -11/3$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1 \Rightarrow u_{33} = -8/5$$

By (1) & (2)

$$LUX = B$$

...(3)

Put

$$UX = Y \quad \dots(4)$$

where $Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

By (3) & (4)

$$LY = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\Rightarrow u = 4$$

$$\frac{2}{3}u + v = 5 \Rightarrow v = 7/3$$

$$\text{and } u + \frac{6}{5}v + w = 7 \Rightarrow w = 1/5$$

By (4)

$$UX = Y$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

 $\Rightarrow$

$$3x + 2y + 7z = 4 \quad \dots(6)$$

$$\frac{5}{3}y - \frac{11}{3}z = \frac{7}{3} \text{ or } 5y - 11z = 7 \quad \dots(7)$$

$$-\frac{5}{8}z = \frac{1}{5} \Rightarrow z = -1/8$$

$$\text{Put } z = -1/8 \text{ in (7)} \Rightarrow y = 9/8$$

$$\text{and put } y = 9/8, z = -1/8 \text{ in (6)} \Rightarrow x = 7/8$$

$$\text{Ans. } x = 7/8, y = 9/8, z = -1/8$$

Q.7. Apply LU decomposition method to solve the equation :

$$x_1 + 2x_2 + 3x_3 = 14, 2x_1 + 5x_2 + 2x_3 = 18, 3x_1 + x_2 + 5x_3 = 20$$

Sol. : Its matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

 $\Rightarrow$

$$AX = B$$

...(1)

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $B = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$

Put

$$A = LU$$

...(2)

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

On Comparison

$$u_{11} = 1, u_{12} = 2, u_{13} = 3$$

$$l_{21}u_{11} \Rightarrow l_{21} = 2$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31} = 3$$

$$l_{21}u_{12} + u_{22} = 5 \Rightarrow u_{22} = 1$$

$$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow l_{32} = -5$$

$$l_{21}u_{13} + u_{23} = 2 \Rightarrow u_{23} = -4$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 5 \Rightarrow u_{33} = -24$$

$$\text{By (1) \& (2)} \quad LUX = B$$

$$\text{Put} \quad UX = Y$$

$$\text{where} \quad Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\Rightarrow LY = B$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 14 \\ 18 \\ 20 \end{bmatrix}$$

$$u = 14$$

$$2u + v = 18 \Rightarrow v = -10$$

$$3u - 5v + w = 20 \Rightarrow w = -72$$

Now by (4)

$$UX = Y$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -24 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -10 \\ -72 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 + 3x_3 = 14 \quad \dots(5)$$

$$x_2 - 4x_3 = -10 \quad \dots(6)$$

$$-24x_3 = -72 \Rightarrow x_3 = 3$$

$$\text{Put in (6)} \quad x_2 = 2$$

$$\text{and by (5)} \quad x_1 = 1$$

$$\text{Ans. } x_1 = 1, x_2 = 2, x_3 = 3$$

Cholesky-Decomposition Method [Square Root Method]

Let the matrix form of the system of equation be

$$AX = B \quad \dots(1)$$

where A be symmetric matrix

Put

$$A = LL^T \quad \dots(2)$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

By (1) \& (2)

$$LL^T X = B \quad \dots(3)$$

Put

$$L^T X = Y \quad \dots(4)$$

when

$$Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

By (3) \& (4)

$$LY = B \quad \dots(5)$$

and find the values of unknown by eq. (4)

Q.8. Solve the system of linear equations :

$$4x_1 + 2x_2 - 2x_3 = 4, 2x_1 + 10x_2 + 2x_3 = 14, -2x_1 + 2x_2 + 3x_3 = 3 \text{ using the cholesky method.}$$

Sol. : Its matrix form

$$\begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 3 \end{bmatrix}$$

$$AX = B \quad \dots(1)$$

$$\text{when } A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 14 \\ 3 \end{bmatrix}$$

Put

$$A = LL^T \quad \dots(2)$$

Where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\therefore \text{By (2)} \quad \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\text{Or} \quad \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$$\text{On Comparison } l_{11}^2 = 4 \Rightarrow l_{11} = 2$$

$$l_{21}l_{11} = 2 \Rightarrow l_{21} = 1$$

$$l_{31}l_{11} = -2 \Rightarrow l_{31} = -1$$

$$l_{31}l_{11} = -2 \Rightarrow l_{31} = -1$$

$$l_{21}^2 + l_{22}^2 = 10 \Rightarrow l_{22} = 3$$

$$l_{31}l_{21} + l_{32}l_{22} = 2 \Rightarrow l_{32} = 1$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 3 \Rightarrow l_{33} = 1$$

By (1) & (2)

$$LL^T X = B$$

Put

$$L^T X = Y$$

By (3) & (4)

$$LY = B$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 3 \end{bmatrix}$$

$$2u = 4 \Rightarrow u = 2$$

$$u + 3v = 14 \Rightarrow v = 4$$

and

$$-u + v + w = 3 \Rightarrow w = 1$$

By (4)

$$L^T X = Y$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

 $\Rightarrow$

$$2x_1 + x_2 - x_3 = 2$$

$$3x_2 + x_3 = 4$$

$$x_3 = 1$$

Put in (6) $\Rightarrow x_2 = 1$

By (5)

$$x_1 = 1$$

Ans. $x_1 = 1, x_2 = 1, x_3 = 1$

Q.9. Solve the system of linear equations:

$$x + 2y + 3z = 5, 2x + 8y + 22z = 6, 3x + 2y + 82z = -10$$

Using the cholesky method:

Sol: Its matrix form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$AX = B$$

Where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

Put

$$A = LL^T$$

... (1)

Where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

By (2)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

On Comparing

$$l_{11}^2 = 1 \Rightarrow l_{11} = 1$$

$$l_{21}l_{11} = 2 \Rightarrow l_{21} = 2$$

$$l_{31}l_{11} = 3 \Rightarrow l_{31} = 3$$

$$l_{21}^2 + l_{22}^2 = 8 \Rightarrow l_{22} = 2$$

$$l_{31}l_{21} + l_{32}l_{22} = 22 \Rightarrow l_{32} = 8$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82 \Rightarrow l_{33} = 3$$

By (1) & (2)

$$LL^T X = B$$

$$\text{Put } L^T X = Y$$

... (3)

... (4)

Where

$$Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

By (3) & (4)

$$LY = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ -10 \end{bmatrix}$$

$$u = 5$$

$$2u + 2v = 6 \Rightarrow v = -2$$

$$3u + 8v + 3w = -10 \Rightarrow w = -3$$

Now by (4)

$$L^T X = Y$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$$

$$x + 2y + 3z = 5$$

$$2y + 8z = -2$$

$$3z = -3 \Rightarrow z = -1$$

$$2y + 8(-1) = -2 \Rightarrow y = 3$$

$$\text{By (5)} \quad x + 2(3) + 3(-1) = 5 \Rightarrow x = 2$$

$$\text{Ans. } x = 2, y = 3, z = -1$$

... (5)

... (6)

Iterative Method**(1) Jacobi-Iterative Method**

Let the system of linear equation be

$$ax + by + cz = d \quad \dots(1)$$

$$a'x + b'y + c'z = d' \quad \dots(2)$$

$$a_1x + b_1y + c_1z = d_1 \quad \dots(3)$$

By (1)

$$x = \frac{d - dy - cz}{a} \quad \dots(A)$$

$$\text{By (2)} \quad y = \frac{d' - a'x - c'z}{b'} \quad \dots(B)$$

$$z = \frac{d_1 - a_1x - b_1y}{c_1} \quad \dots(C)$$

Step I. (1) Put $y = 0, z = 0$ in (A)

$$x = x^{(1)} \quad \text{be first approximation}$$

(2) Put $x = 0, z = 0$ in (B)

$$y = y^{(1)} \quad \text{be first approximation}$$

(3) Put $x = 0, y = 0$ in (C)

$$z = z^{(1)} \quad \text{be first approximation}$$

Step II - (1) Put $y = y^{(1)}, z = z^{(1)}$ in (A)

$$x = x^{(2)} \quad \text{be second approximation}$$

(2) Put $x = x^{(1)}, z = z^{(1)}$ in (B)

$$y = y^{(2)} \quad \text{be second approximation}$$

(3) Put $x = x^{(1)}, y = y^{(1)}$ in (C)

$$z = z^{(2)} \quad \text{be second approximation}$$

The above process is repeated till the two successive approximation are nearly equal.

Q.10. Solve the following system of equations by Jacobi's**Iterative Method :**

$$10x - 5y - 2z = 3, 4x - 10y + 3z = -3, x + 6y + 10z = -3$$

Sol. : Given that

$$10x - 5y - 2z = 3 \quad \dots(1)$$

$$4x - 10y + 3z = -3 \quad \dots(2)$$

$$x + 6y + 10z = -3$$

By (1)

$$x = \frac{3 + 5y + 2z}{10} \quad \dots(A)$$

$$\text{By (2)} \quad y = \frac{4x + 3z + 3}{10} \quad \dots(B)$$

$$\text{By (3)} \quad z = \frac{-3 - x - 6y}{10} \quad \dots(C)$$

Step I (1) Put $y = 0 = z$ in (A) $\Rightarrow x^{(1)} = 0.3$ (2) Put $x = z = 0$ in (B) $\Rightarrow y^{(1)} = 0.3$ (3) Put $x = y = 0$ in (C) $\Rightarrow z^{(1)} = -0.3$ **Step II** - (1) Put $y = 0.3$ and $z = 0.3$ in (A)

$$x^{(2)} = \frac{3 + 5(0.3) + 2(-0.3)}{10} = 0.39$$

(2) Put $x = 0.3, z = -0.3$ in (B)

$$y^{(2)} = \frac{4(0.3) + 3(-0.3) + 3}{10} = 0.33$$

(3) Put $x = 0.3, y = 0.3$ in (C) $z^{(2)} = \frac{-3 - 0.3 - 6(0.3)}{10} = -0.51$ **Step III** (1) Put $y = 0.33$ and $z = -0.51$ in (A)

$$x^{(3)} = \frac{3 + 5(0.33) + 2(-0.51)}{2} = 0.363$$

(2) Put $x = 0.39, z = -0.51$ in (B)

$$y^{(3)} = \frac{4(0.39) + 3(-0.51) + 3}{10} = 0.303$$

(3) Put $x = 0.39, y = 0.33$ in (C)

$$z^{(3)} = \frac{-3 - 0.39 - 6(0.33)}{10} = -0.537$$

Step IV : (1) Put $y = 0.303, z = -0.537$ in (A)

$$x^{(4)} = \frac{3 + 5(0.303) + 2(-0.537)}{10} = 0.344$$

(2) Put $x = 0.363$ and $z = -0.537$ in (B)

$$y^{(4)} = \frac{3 + 4(0.363) + 3(-0.537)}{10} = 0.2841$$

(3) Put $x = 0.363, y = 0.303$ in (C)

$$z = \frac{-3 - 0.363 - 6(0.303)}{10} = -0.5181$$

Step V (1) Put $y = 0.2841, z = -0.5181$ in (A)

$$x^{(5)} = \frac{3 + 5(0.2841) + 2(-0.5181)}{10} = 0.33843$$

(2) Put $x = 0.3441$, $z = -0.5181$ in (B)

$$y^{(5)} = \frac{3 + 4(0.3441) + 3(-0.5181)}{10} = 0.2822$$

Put $x = 0.3441$, $y = 0.2841$ in (C)

$$z^{(5)} = \frac{-3 - 0.3441 - 6(0.2841)}{10} = -0.50487$$

Step VI (1) Put $y = 0.2822$, $z = -0.50487$ in (A)

$$x^{(6)} = \frac{3 + 5(0.2822) + 2(-0.50487)}{10} = 0.340126$$

(2) Put $x = 0.33843$, $z = -0.50487$ in (B)

$$y^{(6)} = \frac{3 + 4(0.33843) + 3(-0.50487)}{10} = 0.283911$$

Put $x = 0.33843$, $y = 0.2822$ in (C)

$$z^{(6)} = \frac{-3 - 0.33843 - 6(0.2822)}{10} = -0.503163$$

Step VII (1) Put $y = 0.283911$ and $z = -0.503163$ in (A)

$$x^{(7)} = 0.34132$$

(2) Put $z = -0.503163$ and $x = 0.340126$ in (B)

$$y^{(7)} = 0.2851$$

(B) Put $x = 0.340126$ and $y = 0.283911$ in (C)

$$z^{(7)} = -0.50435$$

$\therefore x^{(6)} \equiv x^{(7)} \Rightarrow x = 0.3401$ be an approximation root

$\therefore y^{(6)} \equiv y^{(7)} \Rightarrow 0.2851$ be an approximate root

$\therefore z^{(6)} \equiv z^{(7)} \Rightarrow z = -0.5043$ be an approximate root

Q.11. Solve the following system by Jacobi Iterative Method :

$$10x_1 + 2x_2 + x_3 = 9, \quad 2x_1 + 20x_2 - 2x_3 = -44, \quad -2x_1 + 3x_2 + 10x_3 = 22$$

Sol : Given that

$$10x_1 + 2x_2 + x_3 = 9 \quad \dots(1)$$

$$2x_1 + 20x_2 - 2x_3 = -44 \quad \dots(2)$$

$$-2x_1 + 3x_2 + 10x_3 = 22 \quad \dots(3)$$

By (1)

$$x_1 = \frac{9 - 2x_2 + x_3}{10} \quad \dots(A)$$

$$\text{By (2)} \quad x_2 = \frac{-44 - 2x_1 + 2x_3}{20} \quad \dots(B)$$

$$\text{By (3)} \quad x_3 = \frac{22 + 2x_1 - 3x_2}{10} \quad \dots(C)$$

Step I : (1) Put $y = z = 0$ in (A)

$$x_1^{(1)} = \frac{9 - 0 + 0}{10} = 0.9$$

(2) Put $x = 0 = z$ in (B)

$$x_2^{(1)} = \frac{-44 - 0 + 0}{20} = -2.2$$

(3) Put $x = y = 0$ in (C) $x_3^{(1)} = \frac{22 + 0 - 0}{10} = 2.2$

Step II - (1) Put $x_2 = -2.2$, $x_3 = 2.2$ in (A)

$$x_1^{(2)} = \frac{9 - 2(-2.2) - 2.2}{10} = 1.12$$

(2) Put $x_1 = 0.9$, $x_3 = 2.2$ in (B)

$$x_2^{(2)} = \frac{-44 - 2(0.9) + 2(2.2)}{20} = -2.07$$

(3) Put $x_1 = 0.9$, $x_2 = -2.2$ in (C)

$$x_3^{(2)} = \frac{22 + 2(0.9) - 3(-2.2)}{10} = 3.04$$

Step III : (1) Put $x_2 = -2.07$ and $x_3 = 3.04$ in (A)

$$x_1^{(3)} = \frac{9 - 2(-2.07) - 3.04}{10} = 1.01$$

(2) Put $x_1 = 1.12$, $x_3 = 3.04$ in (B)

$$x_2^{(3)} = \frac{-44 - 2(1.12) + 2(3.04)}{20} = -2.08$$

(3) Put $x_1 = 1.12$ and $x_2 = -2.07$ in (C)

$$x_3^{(3)} = \frac{22 + 2(1.12) - 3(3.04)}{10} = 3.045$$

Step IV : (1) Put $x_2 = -2.08$, $x_3 = 3.045$ in (A)

$$x_1^{(4)} = \frac{9 - 2(-2.08) - 3.045}{10} = 0.997$$

(2) Put $x_1 = 1.01$, $x_3 = 3.045$ in (B)

$$x_2^{(4)} = \frac{-44 - 2(1.01) + 2(3.045)}{20} = -1.997$$

(3) Put $x_1 = 1.01$, $x_2 = -2.08$ in (C)

$$x_3^{(4)} = \frac{22 + 2(1.01) - 3(3.045)}{10} = 3.004$$

Step V (1) Put $x_2 = -1.997$, $x_3 = 3.004$ in (A)

$$x_1^{(5)} = \frac{9 - 2(-1.997) - 3.004}{10} = 0.999$$

(2) Put $x_1 = 0.997$, $x_3 = 3.004$ in (B)

$$x_2^{(5)} = \frac{-44 - 2(0.997) + 2(3.004)}{20} = -1.999$$

(3) Put $x_1 = 0.997$ and $x_2 = -1.997$ in (C)

$$x_3^{(5)} = \frac{22 + 2(0.997) - 3(-1.997)}{10} = 2.999$$

$$x_1^{(4)} \equiv x_1^{(5)}, x_2^{(4)} \equiv x_2^{(5)}, x_3^{(4)} \equiv x_3^{(5)}$$

$$x_1 = 0.999$$

$$x_2 = -1.997$$

$$x_3 = 2.999$$

Q.12. Solve : $5x - y - z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$ by Jacob's Iteration Method :

Sol : Given that

$$5x - y - z = 10 \quad \dots(1)$$

$$2x + 4y = 12 \quad \dots(2)$$

$$x + y + 5z = -1 \quad \dots(3)$$

$$\text{By (1)} \quad x = \frac{10 + y + z}{5} \quad \dots(A)$$

$$\text{By (2)} \quad y = \frac{12 - 2x}{4} = \frac{6 - x}{2} \quad \dots(B)$$

$$\text{By (3)} \quad z = \frac{-1 - x - y}{5} \quad \dots(C)$$

Step I : (1) Put $y = 0$, $z = 0$ in (A)

$$x^{(1)} = \frac{10 + 0 + 0}{5} = 2$$

(2) Put $x = 0$, $z = 0$ in (B)

$$y^{(1)} = \frac{6 - 0}{2} = 3$$

(2) Put $x = 0$, $y = 0$ in (C)

$$z^{(1)} = \frac{-1}{5} = -0.2$$

Step II (1) Put $y = 3$ and $z = -0.2$ in (A)

$$x^{(2)} = \frac{10 + 3 - (-0.2)}{5} = 2.64$$

(2) Put $x = 2$ and $z = -0.2$ in (B)

$$y^{(2)} = \frac{6 - 2}{2} = 2$$

(3) Put $x = 2$, $y = 3$ in (C)

$$z^{(2)} = \frac{-1 - 2 - 3}{5} = -1.2$$

Step III (1) Put $y = 2$ and $z = -1.2$ in (A)

$$x^{(3)} = \frac{10 + 2 - (-1.2)}{5} = 2.64$$

(2) Put $x = 2.64$, $z = -1.2$ in (B)

$$y^{(3)} = \frac{6 - 2.64}{2} = 1.68$$

(3) Put $x = 2.64$, $y = 2$ in (C)

$$z^{(3)} = \frac{-1 - 2.64 - 2}{5} = -1.128$$

Step IV (1) Put $y = 1.68$ and $z = -1.128$ in (A)

$$x^{(4)} = \frac{10 + 1.68 - (-1.128)}{5} = 2.5616$$

(2) Put $z = -1.128$, $x = 2.64$ in (B)

$$y^{(4)} = \frac{6 - 2.64}{2} = 1.68$$

(3) Put $x = 2.64$, $y = 1.68$ in (C)

$$z^{(4)} = \frac{-1 - 2.64 - 1.68}{5} = -1.064$$

Step V (1) Put $y = 1.68$, $z = -1.064$ in (A)

$$x^{(5)} = \frac{10 + 1.68 - (-1.064)}{5} = 2.5488$$

(2) Put $x = 2.5616$, $z = -1.064$ in (B)

$$y^{(5)} = \frac{6 - 2.5616}{2} = 1.7192$$

(3) Put $x = 2.5616$, $y = 1.68$ in (C)

$$z^{(5)} = \frac{-1 - 2.5616 - 1.68}{5} = -1.04232$$

Step VI (1) Put $y = 1.7192$, $z = -1.04232$ in (A)

$$x^{(6)} = \frac{10 + 1.7192 - (-1.04232)}{5} = 2.552304$$

(2) Put $z = -1.04232$, $x = 2.5488$ in (B)

$$y^{(6)} = \frac{6 - 2.5488}{2} = 1.7256$$

(3) Put $x = 2.5488$, $y = 1.7192$ in (C)

$$z^{(6)} = \frac{-1 - 2.5488 - 1.7192}{5} = -1.0536$$

Step VII (1) put $y = 1.7256$, $z = -1.0536$ in (A)

$$x^{(7)} = \frac{10 + 1.7256 - (-1.0536)}{5} = 2.5558$$

(2) Put $z = -1.0536$, $x = 2.552304$ in (B)

$$y^{(7)} = \frac{6 - 2.552304}{2} = 1.7238$$

(3) Put $x = 2.552304$, $y = 1.7256$ in (C)

$$z^{(7)} = \frac{-1 - 2.552304 - 1.7256}{5} = 1.0555$$

$$\text{since } x^{(5)} \equiv x^{(7)}, y^{(5)} \equiv y^{(7)}, z^{(5)} \equiv z^{(7)}$$

$$\therefore x = 2.5558, y = 1.7238, z = 1.0555$$

Q.13. Solve the following equations by Jacobi-Iterative Method :

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Sol : By given equations

$$x = \frac{17 - y + 2z}{20} \quad \dots(A)$$

$$y = \frac{z - 3x - 18}{20} \quad \dots(B)$$

$$z = \frac{25 - 2x + 3y}{20} \quad \dots(C)$$

Step I (1) Put $y = 0 = z$ in (A)

$$x^{(1)} = \frac{17}{20} = 0.85$$

(2) Put $x = z = 0$ in (B)

$$y^{(1)} = \frac{-18}{20} = -0.9$$

(3) Put $x = y = 0$ in (C)

$$z^{(1)} = \frac{25 - 0 - 0}{20} = 1.25$$

Step II. (1) Put $y = -0.9$, $z = 1.25$ in (A)

$$x^{(2)} = \frac{17 - (-0.9) + 2(1.25)}{20} = 1.02$$

(2) Put $x = 0.85$, $z = 1.25$ in (B)

$$y^{(2)} = \frac{1.25 - 3(0.85) - 18}{20} = -0.965$$

(3) Put $x = 0.85$, $y = -0.9$ in (C)

$$z^{(2)} = \frac{25 - 2(0.85) + 3(-0.9)}{20} = 1.03$$

Step III (1) Put $y = -0.965$, $z = 1.03$ in (A)

$$x^{(3)} = \frac{17 - (-0.965) + 2(1.03)}{20} = 1.00125$$

(2) Put $z = 1.03$ and $x = 1.02$ in (B)

$$y^{(3)} = \frac{1.03 - 3(1.02) - 18}{20} = -1.0015$$

(3) Put $x = 1.02$, $y = -0.965$ in (C)

$$z^{(3)} = \frac{25 - 2(1.02) + 3(-0.965)}{20} = 1.00325$$

Step IV (1) Put $y = -1.0015$, $z = 1.00325$ in (A)

$$x^{(4)} = \frac{17 - (-1.0015) + 2(1.00325)}{20} = 1.0004$$

(2) Put $x = 1.00125$, $z = 1.00325$ in (B)

$$y^{(4)} = \frac{1.00325 - 3(1.00125) - 18}{20} = -1.000025$$

(3) Put $x = 1.00125$ and $y = -1.0015$ in (C)

$$z^{(4)} = \frac{25 - 2(1.00125) + 3(-1.0015)}{20}$$

$$z^{(4)} = 0.9965$$

Step V (1) Put $y = -1.00025$, $z = 0.9965$ in (A)

$$x^{(5)} = \frac{17 - (-1.000025) + 2(1.09965)}{20} = 0.999966$$

(2) Put $x = 1.0004$, $z = 0.9965$ in (B)

$$y^{(5)} = \frac{0.9965 - 3(1.0004) - 18}{20} = -1.000078$$

(3) Put $x = 1.0004$, and $y = -1.000025$ in (C)

$$z^{(5)} = \frac{25 - 2(1.0004) + 3(-1.000025)}{20} = 0.999965$$

since $x^{(4)} \equiv x^{(5)}$, $y^{(4)} \equiv y^{(5)}$, $z^{(4)} \equiv z^{(5)}$

Ans. $\therefore x = 0.999966$, $y = -1.000078$, $z = 0.999965$

Gauss-Seidel Iterative Method

Let

$$ax + by + cz = d$$

$$a'x + b'y + c'z = d'$$

$$a_1x + b_1y + c_1z = d_1$$

be a system of linear equations

$$\text{By (1)} \quad x = \frac{d - by - cz}{a} \quad \dots(A)$$

$$\text{By (2)} \quad y = \frac{d' - a'x - c'z}{b'} \quad \dots(B)$$

$$\text{By (3)} \quad z = \frac{d_1 - a_1x - b_1y}{c_1} \quad \dots(C)$$

This method is a modification of the Jacobi-Iterative Method.

Step I (1) Put $y = z = 0$ in (A)

$$x = x^{(1)}$$

be first approximation

(2) Put $z = 0$, $x = x^{(1)}$ in (B)

$$y = y^{(1)} \text{ in (C)}$$

(3) Put $x = x^{(1)}$, $y = y^{(1)}$ in (C)

$$z = z^{(1)}$$

be first approximation

Step II (1) Put $y = y^{(1)}$, $z = z^{(1)}$ in (A)

$$x = x^{(2)} \text{ and } z = z^{(1)} \text{ in (B)}$$

$$y = y^{(2)}$$

be second approximation

(3) Put $x = x^{(2)}$ and $y = y^{(2)}$ in (C)

$$z = z^{(2)}$$

be second approximation

The above process is repeated till the two successive approximation are nearly equal.

Q.14. Apply Gauss-Seidal method to solve the equations :
 $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$

Sol: By given equation

$$x = \frac{17 - y + 2z}{20} \quad \dots(A)$$

$$y = \frac{z - 3x - 18}{20} \quad \dots(B)$$

$$z = \frac{25 - 2x + 3y}{20} \quad \dots(C)$$

&

Step I (1) Put $y = z = 0$ in (1)

$$x^{(1)} = \frac{17}{20} = 0.85$$

(2) Put $z = 0$, $x = 0.85$ in (B)

$$y^{(1)} = \frac{0 - 3(0.85) - 18}{20} = -1.0275$$

(3) Put $x = 0.85$, $y = -1.0275$ in (C)

$$z^{(1)} = \frac{25 - 2(0.85) + 3(-1.0275)}{20} = 1.0109$$

Step II (1) Put $y = -1.0275$ and $z = 1.0109$ in (A)

$$x^{(2)} = \frac{17 - (-1.0275) + 2(1.0109)}{20} = 1.0025$$

(2) Put $z = 1.0109$, $x = 1.0025$ in (B)

$$y^{(2)} = \frac{1.0109 - 3(1.0025) - 18}{20} = -0.9998$$

(3) Put $x = 1.0025$, $y = -0.9998$ in (C)

$$z^{(2)} = \frac{25 - 2(1.0025) + 3(-0.9998)}{20} = 0.9998$$

Step III (1) Put $y = -0.9998$, $z = 0.9998$ in (A)

$$x^{(3)} = \frac{17 - (-0.9998) + 2(0.9998)}{20} = 0.99997$$

(2) Put $x = 0.99997$, $z = 0.9998$ in (B)

$$y^{(3)} = \frac{0.9998 - 3(0.99997) - 18}{20} = -1.0000055$$

(3) Put $x = 0.99997$, $y = -1.0000055$ in (C)

$$z^{(3)} = \frac{25 - 2(0.99997) + 3(-1.0000055)}{20} = 1.0000075$$

Step IV (1) Put $y = -1.0000055$, $z = 1.0000075$ in (A)

$$x^{(4)} = \frac{17 - (-1.0000055) + 2(1.0000075)}{20} = 1.00000$$

(2) Put $z = 1.0000075$, $x = 1.00000$ in (B)

$$y^{(4)} = \frac{1.00000 - 3(1.0000075) - 18}{20} = 1.00000$$

(3) Put $x = 1.00000$, $y = 1.00000$ in (C)

$$z^{(4)} = 1.00000$$

$$\therefore x^{(4)} \equiv x^{(3)}, y^{(4)} \equiv y^{(3)} \text{ \& } z^{(4)} \equiv z^{(3)}$$

Ans. $x = 1$, $y = 1$, $z = 1$

Q.15. Use Gauss-Seidel method and solve the system of equation.

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

$$27x + 6y - z = 85$$

Sol. : Given that

$$6x + 15y + 2z = 72 \quad \dots(1)$$

$$x + y + 54z = 110 \quad \dots(2)$$

$$27x + 6y - z = 85 \quad \dots(3)$$

$$\text{By (3)} \quad x = \frac{85 + z - 6y}{27} \quad \dots(A) \because |27| > |6| + |1|$$

$$\text{By (1)} \quad y = \frac{72 - 6x - 2z}{15} \quad \dots(B) \because |15| > |6| + |2|$$

$$\text{By (2)} \quad z = \frac{110 - x - y}{54} \quad \dots(C) \because |54| > |1| + |1|$$

Step I (1) Put $y = z = 0$ in (A)

$$x^{(1)} = \frac{85 + 0 - 0}{27} = 3.15$$

(2) Put $z = 0$, $x = 3.15$ in (B)

$$y^{(1)} = \frac{72 - 6(3.15)}{15} = 3.54$$

(3) Put $x = 3.15$, $y = 3.54$ in (C)

$$z^{(1)} = \frac{110 - 3.15 - 3.54}{54} = 1.9131$$

Step II (1) Put $y = 3.54$, $z = 1.9131$ in (A)

$$x^{(2)} = \frac{85 + 1.9131 - 6(3.54)}{27} = 2.4323$$

(2) Put $z = 1.9131$, $x = 2.4323$ in (B)

$$y^{(2)} = \frac{72 - 6(2.4324) - 2(1.9131)}{15} = 3.572$$

(3) Put $x = 2.4323$ and $y = 3.572$ in (C)

$$z^{(2)} = \frac{110 - 2.4323 - 3.572}{54} = 1.9258$$

Step III (1) Put $y = 3.572$, $z = 1.9258$ in (A)

$$x^{(3)} = \frac{85 + 1.9258 - 6(3.572)}{27} = 2.4257$$

(2) Put $x = 2.4257$ and $z = 1.9258$ in (B)

$$y^{(3)} = \frac{72 - 6(2.4257) - 2(1.9258)}{15} = 3.5729$$

(3) Put $x = 2.4257$, $y = 3.5729$ in (C)

$$z^{(3)} = \frac{110 - 2.4257 - 3.5729}{54} = 1.9260$$

Step IV (1) Put $y = 3.5729$ and $z = 1.9260$ in (A)

$$x^{(4)} = \frac{85 + 1.926 - 6(3.5729)}{27} = 2.426$$

(2) Put $z = 1.926$, $x = 2.426$ in (B)

$$y^{(4)} = \frac{72 - 6(2.426) - 2(1.926)}{15} = 3.573$$

(3) Put $x = 2.426$, $y = 3.573$ in (C)

$$z^{(4)} = \frac{110 - 2.426 - 3.573}{54} = 1.926$$

$$\therefore x^{(3)} \equiv x^{(4)}, y^{(3)} \equiv y^{(4)} \text{ and } z^{(3)} \equiv z^{(4)}$$

$$\therefore x = 2.426, y = 3.573, z = 1.926$$

Q.16. Solve the following system of equations by Gauss-Seidal method.

$$28x + 4y - z = 32, \quad x + 3y + 10z = 24, \quad 2x + 17y + 4z = 35$$

Sol. : Given that

$$28x + 4y - z = 32 \quad \dots(1)$$

$$x + 3y + 10z = 24 \quad \dots(2)$$

$$2x + 17y + 4z = 35 \quad \dots(3)$$

By (1)

$$x = \frac{32 + z - 4y}{28} \quad \dots(A) \quad \because |28| > |4| + |-1|$$

By (3)

$$y = \frac{35 - 2x - 4z}{17} \quad \dots(B) \quad \because |17| > |2| + |4|$$

By (2)

$$z = \frac{24 - x - 3y}{10} \quad \dots(C) \quad \because |10| > |1| + |3|$$

Step I (1) Put $y = z = 0$ in (A)

$$x^{(1)} = \frac{32 + 0 - 0}{28} = 1.4285$$

(2) Put $z = 0$, $x = 1.4285$ in (B)

$$y^{(1)} = \frac{35 - 2(1.4285) - 0}{17} = 1.92437$$

(3) Put $x = 1.4285$, $y = 1.92437$ in (C)

$$z^{(1)} = \frac{24 - 1.4285 - 3(1.92437)}{10} = 1.7084$$

Step II (1) Put $y = 1.92437$ and $z = 1.7084$ in (A)

$$x^{(2)} = 0.92896$$

(2) Put $x = 0.92896$ and $z = 1.7084$ in (B)

$$y^{(2)} = 1.54756$$

(3) Put $x = 0.92896$, $y = 1.54756$ in (C)

$$z^{(2)} = 1.84284$$

Step III (1) Put $y = 1.54756$, $z = 1.84284$ in (A)

$$x^{(3)} = 0.98759$$

(2) Put $z = 1.84284$ and $x = 0.98759$ in (B)

$$y^{(3)} = 1.50903$$

(3) Put $x = 0.98759$, $y = 1.54756$ in (C)

$$z^{(3)} = 1.84853$$

Step IV (1) Put $y = 1.50903$, $z = 1.84853$ in (A)

$$x^{(4)} = 0.9933$$

(2) Put $x = 0.9933$, $z = 1.84853$ in (B)

$$y^{(4)} = 1.50702$$

(3) Put $x = 0.9933$, $y = 1.50702$ in (C)

$$z^{(4)} = 1.84856$$

Step V (1) Put $y = 1.50702$, $z = 1.84856$ in (A)

$$x^{(5)} = 0.99359$$

(2) Put $z = 1.84856$, $x = 0.99359$ in (B)

$$y^{(5)} = 1.50698$$

(3) Put $y = 1.50698$, $x = 0.99359$ in (C)

$$z^{(5)} = 1.84855$$

$$\therefore x^{(4)} \approx x^{(5)}, y^{(4)} \approx y^{(5)}, z^{(4)} \approx z^{(5)}$$

$$\therefore x = 0.9935, y = 1.5069, z = 1.5069$$

Objectives

- Gauss Elimination method is a :**
 - Direct method
 - Indirect method
 - No method
 - Threeway method
- Gauss-Seidel iterative method was :**
 - Pre-calculated values from present iteration
 - $x_i(0)$ for all i
 - $x_i(1)$ for all i
 - None of them
- Gauss-Jordan method is a :**
 - Three way method
 - Indirect method
 - Direct method
 - Iterative method
- LU decomposition method is also known as :**
 - Doolittle's method
 - Factorization method
 - Crout's triangularization method
 - All above
- Cholesky decomposition method uses when :**
 - Matrix A is square
 - $|A| \neq 0$
 - A is symmetric
 - None of them
- Jacobi-Iteration method is :**
 - Direct method
 - Indirect method
 - No method
 - Three way method
- Gauss-Jordan method is**
 - Improvement on jacobi method
 - Iterative method
 - Direct method
 - All above
- The necessary condition for convergence of a system of equation by relation method, relaxation parameter ω is :**
 - $\omega < 1$
 - $\omega = 1$
 - $\omega \in (0, 2)$
 - None of them
- The zeroth and first iteration of jacobi method of solving $10x_1 + 2x_2 + x_3 = 9$, $2x_1 + 20x_2 - 2x_3 = -44$, $-2x_1 + 3x_2 + 10x_3 = 22$ are :**

- (a) (0, 0, 0) (0.9, 2.2, -2.2) (b) (0, 0, 0) (0, 9, -2.2, 2.2)
 (c) (0, 0, 0) (0.9, -2.2, .22) (d) None of them
10. The zeroth and first iterate of Gauss-Seidel method of $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 10x_2 = 14$ are :
 (a) (1.2, 0, 0) (1.2, 1.06, 0.948) (b) (0, 1.2, 0) (1.2, 1.06, 0.948)
 (c) (0, 0, 1.2) (1.2, 1.06, 0.948) (d) None of them
11. By Jacobi-iterative method if the initial solution of the following system of equation is (0, 0, 0) then the first iterative will be :
 $10x - 5y - 2z = 3$, $4x - 10y + 3z = -3$, $x + 6y + 10z = -3$
 (a) (-0.3, 0.3, 0.3) (b) (0.3, -0.3, 0.3)
 (c) (0.3, 0.3, -0.3) (d) (0.3, -0.3, -0.3)
12. By Jacobi-iteration method the first iterate of equations $27x + 6y - z = 85$, $6x + 15y + 2z = 72$, $x + y + 54z = 110$ are :
 (a) (0, 0, 0) (b) (3.148, 4.8, 2.037)
 (c) (3.148, 0, 0) (d) None of these
13. By Jacobi Iterative method if the initial solution of the following system of equations is (2, 3, 0) the first iterate will be :
 $5x - y + z = 10$, $2x + 4y = 12$, $x + y + 5z = -1$
 (a) (2.6, 2.3, -1.1) (b) (2.6, 2, -1.2)
 (c) (2, 2.6, -1.2) (d) (2, 2.6, -1.4)
14. The first iterate of system of equations by Gauss-Seidal method :
 $5x_1 + 2x_2 - x_3 = 6$, $x_1 + 6x_2 - 3x_3 = 4$, $2x_1 + x_2 + 4x_3 = 7$
 (a) (1.2, 0.467, 1.033) (b) (1.2, 0, 0)
 (c) (1.2, 0.467, 0) (d) None of them
15. The first iterate of the following system of equation by Gauss-Seidal method : $2x_1 - x_2 = 7$, $-x_1 + 2x_2 - x_3 = 1$, $-x_2 + 2x_3 = 1$:
 (a) (2.5, 3.5, 1.25) (b) (3.5, 2.5, 1.625)
 (c) (1.5, 2.5, 1.25) (d) (3.5, 2.25, 1.625)
16. Jacobi-iterative method is :
 (a) $x^{(k+1)} = D^{-1}[b - (L + U)x^{(k)}]$

- (b) $x^{(k+1)} = -(D + L)^{-1} Ux^{(k)} + (D + L)^{-1}b$
 (c) $x^{(k+1)} = D[b + (L + U)x^{(k)}]$
 (d) $x^{(k+1)} = (D + L)^{-1} Ux^{(k)} - (D + L)^{-1}b$
17. Gauss-Seidal Iterative method is :
 (a) $x^{(k+1)} = -D^{-1} [Lx^{(k)} + Ux^{(k)}] + D^{-1}b$
 (b) $x^{(k+1)} = D^{-1} [Lx^{(k)} + Ux^{(k)}] - D^{-1}b$
 (c) $x^{(k+1)} = -D^{-1} [Lx^{(k+1)} + Ux^{(k)}] + D^{-1}b$
 (d) $x^{(k+1)} = -D^{-1} [Lx^{(k)} + Ux^{(k+1)}] + D^{-1}b$
18. In LU decomposition method, the value of L is -
 (a) $\begin{bmatrix} 1_{11} & 0 & 0 \\ l_{21} & 1_{22} & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$
 (c) $\begin{bmatrix} 1_{11} & 0 & 0 \\ l_{21} & 1_{22} & 0 \\ l_{31} & l_{32} & 1_{33} \end{bmatrix}$ (d) None of these
19. In LU-decomposition method, the value of U is :
 (a) $\begin{bmatrix} u_{11} & 0 & 0 \\ u_{21} & u_{22} & 0 \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$ (b) $\begin{bmatrix} u_{11} & 1 & 0 \\ u_{21} & u_{22} & 1 \\ u_{31} & u_{32} & 1 \end{bmatrix}$
 (c) $\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ (d) None of these

Answer

- | | | | | |
|--------|--------|--------|--------|--------|
| 1.(a) | 2.(a) | 3.(c) | 4.(d) | 5.(c) |
| 6.(b) | 7.(d) | 8.(c) | 9.(b) | 10.(a) |
| 11.(c) | 12.(b) | 13.(b) | 14.(a) | 15.(d) |
| 16.(a) | 17.(c) | 18.(b) | 19.(b) | |

□□□

Unit-IV

Numerical Solution of Ordinary Differential Equation
 : 1. Single step methods- (i) Picard, (ii) Taylor's series, (iii) Euler, (iv) Runge-Kutta. 2. Multistep methods- (i) Predictor-corrector, (ii) Modified Euler, (iii) Milne-Simpson.

Picard's Method (Method of Successive integration)

Let $\frac{dy}{dx} = f(x, y)$... (1)

be given equation, and initial condition,
 when $x = x_0 \Rightarrow y = y_0$

then $y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx$ (First approximation)

$y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx$ (Second approximation)

$y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) dx$ (Third approximation)

$\vdots$

and n^{th} approximation is

$y^{(n)} = y_0 + \int_{x_0}^x f(x, y^{(n-1)}) dx$

Q.1. If $\frac{dy}{dx} = x - y$ with $y = 1$ when $x = 0$ then find the value of y when $x = 0.2$ by Picard's method.
 Sol : Given that

$\frac{dy}{dx} = x - y$... (1)

with $x_0 = 0, y_0 = 1$
 By Picard's method, first approximation

$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx$

$y^{(1)} = 1 + \int_0^x [x - 1] dx$

$= 1 + \left[\frac{x^2}{2} - x \right]_0^x$

By (1)

$y^{(1)} = 1 - x + \frac{x^2}{2}$... (2)

$x = 0.2$

When

$y^{(1)} = y(0.2) = 1 - 0.2 + \frac{(0.2)^2}{2}$

$y^{(1)} = y(0.2) = 0.8200$

Second approximation

$y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx$

$y^{(2)} = 1 + \int_0^x \left[x - \left(1 - x + \frac{x^2}{2} \right) \right] dx$

$= 1 + \int_0^x \left[-\frac{x^2}{2} + 2x - 1 \right] dx$

$= 1 + \int_0^x \left[-\frac{x^2}{2} + 2x - 1 \right] dx$

$= 1 + \left[-\frac{x^3}{6} + x^2 - x \right]_0^x$

$y^{(2)} = -\frac{x^3}{6} + x^2 - x + 1$... (3)

$x = 0.2$

When

$y^{(2)} = y(0.2) = -\frac{(0.2)^3}{6} + (0.2)^2 - (0.2) + 1$

$y^{(2)} = y(0.2) = 0.83867$

Third approximation

$y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) dx$

$y^3 = 1 + \int_0^x \left[x - \left(-\frac{x^3}{6} + x^2 - x + 1 \right) \right] dx$

$y^{(3)} = 1 + \int_0^x \left[\frac{x^3}{6} - x^2 + 2x - 1 \right] dx$

$= 1 + \left[\frac{x^4}{24} - \frac{x^3}{3} + x^2 - x \right]_0^x$

$y^{(3)} = \frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1$... (4)

$x = 0.2$

When

$$y^{(3)} = \frac{(0.2)^4}{24} - \frac{(0.2)^3}{3} + (0.2)^2 - (0.2) + 1$$

$$y^{(3)} = y(0.2) = 0.8374$$

Ans

Q.2. Use Picard's method to solve $\frac{dy}{dx} = 1 + xy$; with $x_0 = 2$,

$$y_0 = 0$$

Sol: Given that

$$\frac{dy}{dx} = 1 + xy \quad \dots(1)$$

with $x_0 = 2, y_0 = 1$

By Picard's method, first approximation

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$y^{(1)} = 0 + \int_2^x [1 + x \cdot (0)] dx$$

$$= \int_2^x [1] dx = [x]_2^x$$

$$y^{(1)} = x - 2$$

By (*)

... (1)

Second approximation

$$y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx$$

$$y^{(2)} = 0 + \int_2^x [1 + x(x - 2)] dx$$

By (1)

$$= \int_2^x [1 + x^2 - 2x] dx = \int_2^x [1 - 2x + x^2] dx$$

$$y^{(2)} = \left[x - x^2 + \frac{x^3}{3} \right]_2^x$$

$$= \left[x - x^2 + \frac{x^3}{3} - \left\{ 2 - (2)^2 + \frac{(2)^3}{3} \right\} \right]$$

$$y^{(2)} = x - x^2 + \frac{x^3}{3} - \frac{2}{3}$$

... (2)

Third approximation

$$y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) dx$$

$$= 0 + \int_2^x \left[1 + x \left(x - x^2 + \frac{x^3}{3} - \frac{2}{3} \right) \right] dx \text{ by (2)}$$

$$= \int_2^x \left[1 + x^2 - \frac{x^3}{2} + \frac{x^4}{3} - \frac{2}{3}x \right] dx$$

$$= \left[x + \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{2}{6}x^2 \right]_2^x$$

$$= \left[x + \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{1}{3}x^2 - \left\{ 2 + \frac{(2)^3}{3} - \frac{(2)^4}{8} + \frac{(2)^5}{15} - \frac{1}{3}(2)^2 \right\} \right]$$

$$y^{(3)} = x - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^4}{8} + \frac{x^5}{15} - \frac{52}{15} \quad \dots(3)$$

Q.3. Use Picard's method the approximate y when $x = 0$,

1 given that $\frac{dy}{dx} = x + y$ with $x = 0 \Rightarrow y = 0$.

Sol: Given that

$$\frac{dy}{dx} = x + y \quad \dots(*)$$

with $x_0 = 0, y_0 = 0$
By Picard's method, first approximation

$$y^{(1)} = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$y^{(1)} = 0 + \int_0^x [x + 0] dx$$

By (*)

$$y^{(1)} = \left[\frac{x^2}{2} \right]_0^x = \frac{x^2}{2} \quad \dots(5)$$

$$y^{(1)} = \frac{x^2}{2} \quad \dots(1)$$

Second approximation

$$y^{(2)} = y_0 + \int_{x_0}^x f(x, y^{(1)}) dx$$

$$y^{(2)} = 0 + \int_0^x \left[x + \frac{x^2}{2} \right]$$

By (1)

$$= \left[\frac{x^2}{2} + \frac{x^3}{6} \right]_0^x$$

$$y^{(2)} = \frac{x^3}{6} + \frac{x^2}{2} \quad \dots(2)$$

Third approximation

$$y^{(3)} = y_0 + \int_{x_0}^x f(x, y^{(2)}) dx$$

$$= 0 + \int_0^x \left[x + \frac{x^3}{6} + \frac{x^2}{2} \right] dx$$

$$y^{(3)} = \left[\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^3}{6} \right]_0^x$$

$$y^{(3)} = \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2}$$

Now, by (1), (2) & (3), When $x = a$,

$$y^{(1)} = \frac{(0.1)^2}{2} = 0.005$$

$$y^{(2)} = \frac{(0.1)^3}{6} + \frac{(0.1)^2}{2} = 0.005167$$

and

$$y^{(3)} = \frac{(0.1)^4}{24} + \frac{(0.1)^3}{6} + \frac{(0.1)^2}{2} = 0.00517$$

Taylor's Series

Let $\frac{dy}{dx} = f(x, y)$

be given equation, and initial condition,
when $x = x_0 \Rightarrow y = y_0$

$$y = y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots$$

Q.4. Solve $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.

Sol : Given that

$$y' = \frac{dy}{dx} = 1 - 2xy \quad \dots(1)$$

with $y(0) = 0$ i.e. $x_0 = 0, y_0 = 0$

By Taylor's-Series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 \dots(*)$$

Put

$$x = x_0 = 0 \text{ in } (1)$$

$$(y')_0 = 1 - 2.0.0 = 1$$

by (2)

...(3)

Diff. (1) $y'' = 0 - 2[xy' + y]$...(2)

$$\Rightarrow (y'')_0 = -2[0 + 0] = 0$$

Diff. (2) $y''' = [\{ xy'' + y' \} + y'] = -2[xy'' + 2y']$...(3)

$$(y''')_0 = -2[0 + 2(1)] = -4$$

Diff. (3) $y^{(iv)} = [\{ xy''' + y'' \} + y''] = -2[xy''' + 2y'']$...(4)

$$(y^{(iv)})_0 = -2[0 + 0] = 0$$

Diff. (4) $y^{(v)} = [\{ xy^{(iv)} + y^{(iii)} \} + 3y^{(iii)}]$...(5)

$$y^{(v)} = -2[xy^{(iv)} + 4y^{(iii)}]$$

$$(y^{(v)})_0 = -2[0 + 4(-4)] = 32$$

$$\vdots$$

$\therefore$ By (*), with $x_0 = 0$

$$y = y(x) = 0 + (x - 0)1 + \frac{(x - 0)^2}{2!}0 + \frac{(x - 0)^3}{3!}(-4) + \frac{(x - 0)^4}{4!}0$$

$$+ \frac{(x - 0)^5}{5!}(32) + \dots$$

or $y = x - \frac{2x^3}{3} + \frac{4}{15}x^5 + \dots$

Q.5. Solve $\frac{dy}{dx} = x + y^2, y(0) = 1$, using TAYLOR'S Series and find $y(0.1)$ and $y(0.2)$

Sol : Given that

$$y' = \frac{dy}{dx} = x + y^2 \quad \dots(1)$$

with $y(0) = 1$ i.e. $x_0 = 0, y_0 = 1$

By (1) $(y')_0 = 0 + 1^2 = 1$

Diff. (1) $y'' = 1 + 2yy'$...(2)

$$(y'')_0 = 1 + 2(1)(1) = 3$$

Diff. (2) $y''' = 0 + 2\{yy'' + y^2 y'\}$...(3)

$$(y''')_0 = 2[1(3) + 1] = 2[4] = 8$$

Diff. (3) $y^{(iv)} = 2[\{yy''' + y^2 y''\} + 2y^2 y']$

$$y^{(iv)} = 2[yy''' + 3y^2 y''] \quad \dots(4)$$

$$\Rightarrow (y^{(iv)})_0 = 2[1(8) + 3(1)(3)] = 34$$

$$\vdots$$

Put these values in Taylor's series i.e.

$$y = y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \dots$$

$$\Rightarrow y = y(x) = 1 + (x-0) + \frac{(x-0)^2}{2!} + 3 \frac{(x-0)^3}{3!} + \frac{(x-x_0)^4}{4!} \cdot 3y + \dots$$

$$y = y(x) = 1 + x + \frac{3x^2}{2} + \frac{4}{2}x^3 + \frac{17}{12}x^4 + \dots \quad \dots(A)$$

Put $x = 0.1$

$$y = y(0.1) = 1 + (0.1) + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4 + \dots$$

$$y = y(0.1) = 1.1165$$

Now Put $x = 0.2$ in (A)

$$y = y(0.2) = 1 + (0.2)^2 + \frac{3}{2}(0.2)^2 + \frac{4}{3}(0.2)^3 + \frac{17}{12}(0.2)^4 + \dots$$

$$y = y(0.2) = 1.2729$$

Q.6. Solve the equation $\frac{dy}{dx} = -yx^2$ with $y(0) = 2$ by Taylor's Series.

Sol : Given that $y' = \frac{dy}{dx} = -yx^2 \quad \dots(1)$

with $y(0) = 2$ i.e. $x_0 = 0, y_0 = 2$

By (1) $(y')_0 = 0$

Diff. (1) $y'' = -[2xy + x^2y'] \quad \dots(2)$
 $(y'')_0 = -[0 + 0] = 0$

Diff. (2) $y''' = [2\{xy' + y\} + \{2xy' + x^2y''\}] \quad \dots(3)$
 $y''' = [4xy' + x^2y'' + 2y]$

Diff. (3) $y^{iv} = -[4\{xy'' + y'\} + \{2xy'' + x^2y'''\} + 2y'] \quad \dots(4)$
 $y^{iv} = -[6xy'' + 6y' + x^2y''']$
 $(y^{iv})_0 = [0 + 6 \cdot 0 + 0] = 0$

Diff. (4) $y^v = [6\{xy''' + y''\} + 6y'' + \{x^2y''' + 2xy^{iv}\}] \quad \dots(5)$
 $y^v = -2[8xy^{iv} + 12y''' + x^2y^{iv}]$
 $(y^v)_0 = -[0 + 0 + 0] = 0$

Diff. (5) $y^{vi} = -[8\{xy^{iv} + y'''\} + 12y''' + \{x^2y^{iv} + 2xy^{iv}\}] \quad \dots(6)$
 $y^{vi} = [10xy^{iv} + 20y''' + x^2y^{iv}]$
 $(y^{vi})_0 = -[0 + 20(-4) + 0] = 80$

Put these values in Taylor's series i.e.

$$y = y(x) = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots$$

$$\text{i.e. } y = y(x) = 2 + (x-0) \cdot 0 + \frac{(x-0)^2}{2!} + 3 \frac{(x-0)^3}{3!} + \frac{(x-x_0)^4}{4!} \cdot 0 + \frac{(x-0)^5}{5!} \cdot 0 + \frac{(x-0)^6}{6!} (80) + \dots$$

or $y = y(x) = 2 - \frac{4}{3!}x^3 + \frac{80}{6!}x^4 + \dots$ ■

Ordinary Differential Equation

(Numerical Solution of differential equation)

(1) Euler's Method

Let $\frac{dy}{dx} = f(x, y) \quad \dots(1)$

be differential equation of first order with initial condition $y(x_0) = y_0$ i.e. when $x = x_0 \Rightarrow y = y_0$ with given h and $x_n = x_0 + nh$ then

$$y_1 = y(x_1) = y_0 + hf(x_0, y_0)$$

$$y_2 = y(x_2) = y_1 + hf(x_1, y_1)$$

In general,

$$y_n = y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

(2) Modified Euler's Method

From Euler's Method

$$y^{(1)} = y_1 = y_0 + hf(x_0, y_0)$$

By Modified Euler's Method

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

and $y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$

and so on, then $y_1^{(m)} = y_1^{(m+1)}$

Q.7. Solve the equation $\frac{dy}{dx} = 1 - y$ with initial condition $x = 0, y = 0$ using Euler's method and tabulate the solution at $x = 0.1, 0.2, 0.3$.

Sol : Given that :

$$\frac{dy}{dx} = f(x, y) = 1 - y \quad \dots(*)$$

$x_0 = 0, y_0 = 0$ and $x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$
 $\therefore x_n = x_0 + nh$ then $x_1 = x_0 + h \Rightarrow h = x_1 - x_0 = 0.1$

By Euler's method

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ y_1 &= y(0.1) = 0 + (0.1) [1 - 0] \\ y_1 &= 0.1 \\ \text{and } y_2 &= y_1 + hf(x_1, y_1) \\ y_2 &= y(x_2) = 0.1 + (0.1) [1 - 0.1] \\ \Rightarrow y_2 &= y(0.2) = 0.1 + 0.09 = 0.19 \\ \text{and } y_3 &= y(x_3) = y_2 + hf(x_2, y_2) \\ y_3 &= y(0.3) = 0.19 + (0.1) [1 - 0.19] \\ y_3 &= 0.271 \end{aligned}$$

& Its table

x	0.1	0.2	0.3
y	0.1	0.19	0.271

Q.8. Use Euler's method to find $y(0.4)$ from differential equation $\frac{dy}{dx} = xy, y(0) = 1$.

Take for each step $h = 0.1$

Sol. : Given that

$$\frac{dy}{dx} = f(x, y) = xy \quad \dots (*)$$

& $y(0) = 1$, i.e. $x_0 = 0, y_0 = 1$ & $h = 0.1$

$$x_n = 0.4, \therefore x_n = x_0 + nh$$

$$\Rightarrow x_1 = 0 + 1(0.1) \Rightarrow x_1 = 0.1, \text{ and so, } x_2 = 0.2, x_3 = 0.3 \text{ \& } x_4 = 0.4$$

By Euler's method

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \Rightarrow y_1 &= y(0.1) = 1 + (0.1) [0(1)] \\ y_1 &= y(0.1) = 1 \\ \& y_2 &= y(x_2) = y_1 + hf(x_1, y_1) \\ &= 1 + (0.1) [(0.1)(0.1)(1)] = 1 + 0.01 \\ y_2 &= y(0.2) = 1.01 \\ \text{Now } y_3 &= y(x_3) = y_2 + hf(x_2, y_2) \\ y_3 &= y(0.3) = 1.01 + (0.1) [(0.2)(1.02)] \\ y_3 &= 1.0302 \\ y_4 &= y(0.4) = y(x_4) = y_3 + hf(x_3, y_3) \\ y_4 &= y(0.4) = 1.0302 + (0.1) [(0.3)(1.0306)] \\ y_4 &= 1.061106 \end{aligned}$$

Q.9. Using Euler's method solve the differential equation in six steps $\frac{dy}{dx} = x + y, y(0) = 0$, choosing $h = 0.2$

Sol. Given that $\frac{dy}{dx} = f(x, y) = x + y \quad \dots (*)$

with $y(0) = 0 \Rightarrow$ when $x_0 = 0, y_0 = 0$ & $h = 0.2, h = 0.2$

$$\therefore x_n = x_0 + nh, \therefore x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0, x_6 = 1.2$$

By Euler's method

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \Rightarrow y &= y(0.2) = 0 + (0.2) [0 + 0] \Rightarrow y_1 = y(0.2) = 0 \text{ By } (*) \\ \text{and } y_2 &= y(x_2) = y_1 + hf(x_1, y_1) \\ y_2 &= y(0.4) = 0 + (0.2) [0.2 + 0] = 0.04 \\ \therefore y_3 &= y(x_3) = y_2 + hf(x_2, y_2) \\ y_3 &= y(0.6) = 0.04 + (0.2) [0.4 + 0.04] \text{ By } (*) \\ y_3 &= 0.128 \\ y_4 &= y(x_4) = y_3 + hf(x_3, y_3) \\ &= 0.128 + (0.2) [0.6 + 0.128] \text{ By } (*) \\ y_4 &= y(0.8) = 0.2736 \\ \text{and } y_5 &= y(x_5) = y_4 + hf(x_4, y_4) \\ y_5 &= y(1.0) = 0.2736 + (0.2) [0.8 + 0.2736] \text{ By } (*) \\ y_5 &= y(1.0) = 0.48832 \\ \& \therefore y_6 &= y(x_6) = y_5 + hf(x_5, y_5) \\ y_6 &= y(1.2) = 0.48832 + 0.2(1.0 + 0.48832) \text{ By } (*) \\ y_6 &= y(1.2) = 0.785994 \end{aligned}$$

Q.10. Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$. Find y for $x = 0.1$ by Euler's method (five steps)

Sol. : Given that $\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y) \quad \dots (*)$

with $x_0 = 0, y_0 = 1, x_n = 0.1, n = 5$

$$\therefore x_n = x_0 + nh \Rightarrow h = \frac{x_n - x_0}{n} = \frac{0.1 - 0}{5} = 0.02$$

then $x_1 = 0.02, x_2 = 0.04, x_3 = 0.06, x_4 = 0.08, x_5 = 0.1$

By Euler's method

$$\begin{aligned} y_1 &= y(x_1) = y_0 + hf(x_0, y_0) \\ \Rightarrow y_1 &= y(0.02) = 1 + (0.02) \left[\frac{1-0}{1+0} \right] \text{ By } (*) \end{aligned}$$

$$\begin{aligned}
 & \text{and } y_1 = y(0.02) = 1.02 \\
 & y_2 = y(x_2) = y_1 + hf(x_1, y_1) \\
 & y_2 = y(0.04) = 1.02 + (0.02) \left[\frac{1.02 - 0.02}{1.02 + 0.02} \right] \quad \text{By (*)} \\
 \Rightarrow & y_2 = y(0.04) = 1.0392 \\
 & \& \therefore y_3 = y(x_3) = y_2 + hf(x_2, y_2) \\
 & y_3 = y(0.06) = 1.0392 + (0.02) \left[\frac{1.0392 - 0.04}{1.0392 + 0.04} \right] \quad \text{By (*)} \\
 \Rightarrow & y_3 = y(0.06) = 1.0577 \\
 \text{and } & y_4 = y(x_4) = y_3 + hf(x_3, y_3) \\
 & y_4 = y(0.08) = 1.0577 + (0.02) \left[\frac{1.0577 - 0.06}{1.0577 + 0.06} \right] \quad \text{By (*)} \\
 & y_4 = y(0.08) = 1.0756 \\
 \& \therefore & y_5 = y(x_5) = y_4 + hf(x_4, y_4) \\
 & y_5 = y(0.1) = 1.0756 + (0.02) \left[\frac{1.0756 - 0.08}{1.0756 + 0.01} \right] \\
 & y_5 = y(1.0) = 1.0928 \quad \text{Ans. } \blacksquare
 \end{aligned}$$

Q.11. Find the value of y at $x = 0.1$ for given $\frac{dy}{dx} = x^2 + y$ with $y(0) = 0.94$ take $h = 0.1$ by using modified Euler's method.

Sol. : Given that $\frac{dy}{dx} = f(x, y) = x^2 + y$...(*)

$$x_0 = 0, y_0 = 0.94, h = 0.1 \text{ and } x_1 = 0.1$$

By Euler's method

$$y_1^{(1)} = y_1 = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = 0.94 + (0.1) [0^2 + (0.94)]$$

$$y_1^{(1)} = 1.034$$

Now, by modified Euler's method

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$\Rightarrow y_1^{(2)} = 0.94 + \frac{0.1}{2} [(0^2 + 0.94) + [(0.1)^2 + (1.034)]]$$

$$y_1^{(2)} = 1.0392$$

By (*)

$$\& \therefore y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$\Rightarrow y_1^{(3)} = 0.94 + \frac{0.1}{2} [(0 + 0.94) + (0.1)^2 + 1.0392] \quad \text{By (*)}$$

$$y_1^{(3)} = 1.0395$$

$$\text{Again } y_1^{(4)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$y_1^{(4)} = 0.94 + \frac{0.1}{2} [(0 + 0.94) + [(0.1)^2 + 1.0395]]$$

$$y_1^{(4)} = 1.0395$$

$$\therefore y_1^{(3)} = y_1^{(4)}$$

$$\therefore y_1 = y(0.1) = 1.0395 \quad \blacksquare$$

Q.12. Using modified Euler's method solve $\frac{dy}{dx} = 1 + xy$ with $y(0) = 2, h = 0.1$. Find $y(0.1)$

Sol : Given that

$$\frac{dy}{dx} = f(x, y) = 1 + xy \quad \dots(*)$$

$$x_0 = 0, y_0 = 2, h = 0.1 \text{ and } x_1 = 0.1$$

By Euler's method

$$y_1^{(1)} = y_0 + hf(x_0, y_0)$$

$$= 2 + (0.1) [1 + (0)(2)]$$

$$y_1^{(1)} = 2.1$$

By modified Euler's method

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 2 + \frac{0.1}{2} [[1 + 0(2)] + [1 + (0.1)(2.1)]]$$

$$y_1^{(2)} = 2.1105$$

By (*)

$$\& y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$y_1^{(3)} = 2 + \frac{0.1}{2} [[1 + 0] + [1 + (0.1)(2.1105)]]$$

$$y_1^{(3)} = 2.1106$$

$$\begin{aligned} & \& y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ \Rightarrow & y_1^{(1)} = 2 + \frac{0.1}{2} [(1+0) + (1 + (0.1)(2.1106))] \\ \Rightarrow & y_1^{(1)} = 2.1106 \\ & y_1^{(2)} = y_1^{(1)} \\ & y_1 = y(0.1) = 2.1106 \end{aligned}$$

Q.13. Use modified Euler's method to compute y for $x = 0.05$.

Given that: $\frac{dy}{dx} = x + y$ with initial conditions $x_0 = 0$, $y_0 = 1$

Sol: Given that

$$\frac{dy}{dx} = x + y = f(x, y) \quad \dots (*)$$

with $x_0 = 0$, $y_0 = 1$, $x_1 = 0.05 \Rightarrow h = x_1 - x_0$
 $h = 0.05 - 0 = 0.05$

By Euler's method

$$\begin{aligned} y_1^{(1)} &= y_0 + hf(x_0, y_0) \\ y_1^{(1)} &= 1 + (0.05)[0 + 1] \\ y_1^{(1)} &= 1.05 \end{aligned}$$

By modified Euler's method

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ &= 1 + \frac{0.05}{2} [(0+1) + (0.05+1.05)] \end{aligned}$$

$$\Rightarrow y_1^{(2)} = 1.0525$$

$$\begin{aligned} \& y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ &= 1 + \frac{0.05}{2} [(0+1) + (0.05+1.0525)] \end{aligned}$$

$$y_1^{(3)} = 1.0526$$

$$\& y_1^{(4)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 1 + \frac{0.05}{2} [(0+1) + (0.05+1.0526)]$$

$$y_1^{(4)} = 1.0526$$

$$y_1^{(3)} = y_1^{(4)} \Rightarrow y_1 = y(0.05) = 1.0526 \quad \blacksquare$$

Q.14. Using modified Euler's method, obtain a solution of the equation $\frac{dy}{dx} = 1 - y$ with $y(0) = 0$ in range $0 \leq x \leq 0.2$ by taking $h = 0.1$

Sol: Given that

$$\frac{dy}{dx} = f(x, y) = 1 - y \quad \dots (*)$$

with $x_0 = 0$, $y_0 = 0$, $h = 0.1$ & $x_n = x_0 + nh$

$$\Rightarrow x_1 = 0.1, x_2 = 0.2$$

By Euler's method

$$\begin{aligned} y_1^{(1)} &= y_0 + hf(x_0, y_0) \\ &= 0 + (0.1)[1 - 0] \\ y_1^{(1)} &= 0.1 \end{aligned}$$

Now by modified Euler's method:

$$\begin{aligned} y_1^{(2)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})] \\ &= 0 + \frac{0.1}{2} [(1-0) + (1-0.1)] \\ y_1^{(2)} &= 0.095 \end{aligned}$$

$$\begin{aligned} \& y_1^{(3)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})] \\ &= 0 + \frac{0.1}{2} [(1-0) + (1-0.095)] \\ y_1^{(3)} &= 0.953 \end{aligned}$$

$$\begin{aligned} \text{Again, } y_1^{(4)} &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(3)})] \\ &= 0 + \frac{0.1}{2} [(1-0) + (1-0.953)] \end{aligned}$$

$$y_1^{(4)} = 0.0952$$

Again, $y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$$y_1^{(2)} = 0 + \frac{0.1}{2} [(1 - 0) + (1 - 0.952)]$$

$$y_1^{(2)} = 0.0952$$

$$y_1^{(3)} = y_1^{(2)}$$

$$y_1 = y(0.1) = 0.0952$$

Now, by Euler's method

$$y_1^{(2)} = y(x_2) = y_1 + hf(x_1, y_1)$$

$$y_1^{(2)} = 0.18568$$

Now by modified Euler's method

$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$y_2^{(2)} = 0.0952 + \frac{0.1}{2} [(1 - 0.0952) + (1 - 0.18568)]$$

$$y_2^{(2)} = 0.181166$$

$$\& \quad y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 0.0952 + \frac{0.1}{2} [(1 - 0.0952) + (1 - 0.181166)]$$

$$y_2^{(2)} = 0.1813817$$

$$y_2^{(4)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(3)})]$$

$$= 0.952 + \frac{0.1}{2} [(1 - 0.0952) + (1 - 0.1813817)]$$

$$y_2^{(4)} = 0.1813709$$

$$y_2^{(3)} = y_2^{(4)}$$

$$y_2 = y(0.2) = 0.181370$$

Runge-Kutta Method

Given that

$$\frac{dy}{dx} = f(x, y)$$

with $y(x_0) = y_0$ i.e. when $x = x_0$, $y = y_0$ and h & $x_n = x_0 + nh$

then

(1) Second Order Formula

$$y_1 = y(x_1) = y_0 + \frac{1}{2} (k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$

& $k_2 = f(x_0 + h, y_0 + k_1)$

$$\& \quad y_2 = y(x_2) = y_1 + \frac{1}{2} (k_1 + k_2)$$

when $k_1 = hf(x_1, y_1)$

& $k_2 = hf(x_1 + h, y_1 + k_1)$ and so on

(2) Fourth Order Method

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_1)$$

Q.15. Use Runge Kutta method to approximate, when $x = 0.1$.

Given that $x = 0$ when $y = 1$ and $\frac{dy}{dx} = x + y$

Sol. : Given that $\frac{dy}{dx} = x + y$... (1)

with $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = x_1 - x_0 = 0.1$

By Runge Kutta Method (fourth order)

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4]$$

$$\therefore k_1 = hf(x_0, y_0) = 0.1 [0 + 1] = 0.1 \quad \text{By (1)}$$

$$\& \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)\right]$$

$$\Rightarrow k_2 = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) \left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.11}{2}\right)\right]$$

$$k_3 = 0.1105$$

$$\& \quad k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) [(0 + 0.1) + (1 + 0.1102)]$$

$$k_4 = 0.12105$$

Put these values in (*)

$$y_1 = y(0.1) = 1 + \frac{1}{6} [0.1 + 2(0.11 + 0.1105) + 0.12105]$$

$$y_1 = 1.11034$$

Q.16. Solve $\frac{dy}{dx} = \frac{1}{x+y}$ for $x = 0.5$ by using Runge-Kutta method with $x_0 = 0, y_0 = 1$ (take $h = 0.5$)

Sol. : Given that $\frac{dy}{dx} = \frac{1}{x+y}$... (1)

$$\text{with } x_0 = 0, y_0 = 1, h = 0.5, x_n = x_0 + nh$$

$$\therefore x_1 = x_0 + 1 \cdot h = 0 + 0.5 \Rightarrow x_1 = 0.5$$

By Runge-Kutta method (fourth order)

$$y_1 = y(0.5) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots (*)$$

$$\text{when } k_1 = hf(x_0, y_0) = \frac{0.5}{0+1}$$

$$\Rightarrow k_1 = 0.5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.5 \left[\frac{1}{\frac{0.5}{2} + 1 + \frac{0.5}{2}} \right]$$

$$k_2 = 0.3333$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.5 \left[\frac{1}{\frac{0.5}{2} + 1 + \frac{0.3333}{2}} \right]$$

$$k_3 = 0.35294$$

$$\& \quad k_4 = hf(x_0 + h, y_0 + k_3) = 0.5 \left[\frac{1}{0.5 + 1 + 0.35294} \right]$$

$$k_4 = 0.26984$$

Put these values in (*)

$$y_1 = y(0.5) = 1 + \frac{1}{6} [0.5 + 2(0.3333 + 0.35294) + 0.26984]$$

$$\Rightarrow y_1 = y(0.5) = 1.35705$$

Ans.

Q.17. Use Runge Kutta method to find y when $x = 1.2$ in steps of 0.1 given that $\frac{dy}{dx} = x^2 + y^2, y(1) = 1.5$

Sol. : Given that $\frac{dy}{dx} = x^2 + y^2$... (1)

$$\text{with } x_0 = 1, y_0 = 1.5, h = 0.1$$

$$\therefore x_0 = x_0 + nh \Rightarrow x_1 = 1 + (0.1) = 1.1$$

$$x_2 = 1 + 2(0.1) = 1.2$$

& By Runge Kutta Method (fourth order)

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_0 + 2(k_1 + k_2) + k_4] \quad \dots (*)$$

$$\text{when } k_1 = hf(x_0, y_0) = 0.1[0(1)^2 + (1.5)^2] = 0.325$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1) \left[\left(1 + \left(\frac{0.1}{2}\right)\right)^2 + \left(1.5 + \left(\frac{0.325}{2}\right)\right)^2 \right]$$

$$k_2 = 0.38664$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \left[\left(1 + \frac{0.1}{2}\right)^2 + \left(1.5 + \frac{0.38664}{2}\right)^2 \right]$$

$$\Rightarrow k_3 = 0.39698$$

$$\& \quad k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[(1 + 0.1)^2 + (1.5 + 0.39698)^2]$$

$$\Rightarrow k_4 = 0.48085$$

Put these values in equation (*)

$$y_1 = y(1.1) = 1.5 + \frac{1}{6} [0.325 + 2(0.38664 + 0.39698) + 0.48085]$$

$$y_1 = y(1.1) = 1.895515$$

Again by Runge Kutta Method

$$y_2 = y(x_2) = y_1 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots (**)$$

$$\text{where } k_1 = hf(x_1, y_1) = (0.1) [(1.1)^2 + (1.895515)^2]$$

$$\Rightarrow k_1 = 0.4802977$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 \left[\left(1.1 + \frac{0.1}{2}\right)^2 + \left(1.895515 + \frac{0.4802977}{2}\right)^2 \right]$$

$$\Rightarrow k_2 = 0.58835$$

$$\& k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 \left[\left(1.1 + \frac{0.1}{2}\right)^2 + \left(1.895515 + \frac{0.58835}{2}\right)^2 \right]$$

$$\Rightarrow k_3 = 0.61172$$

$$\& k_4 = hf(x_1 + h, y_1 + k_3) = 0.1[(1.1 + 0.1)^2 + (1.895515 + 0.61172)]$$

$$k_4 = 0.77262$$

Put these values in (**)

$$y_2 = y(1.2) = 1.895515 + \frac{1}{6} [0.4802977 + 2(0.58835 + 0.61172) + 0.77262]$$

$$y_2 = y(1.2) = 2.5043579$$

Ans.

Q.18. Apply Runge Kutta Method of fourth order to solve

$$10 \frac{dy}{dx} = x^2 + y^2, y(0) = 1, \text{ for } x = 0.1$$

Sol. : Given that

$$\frac{dy}{dx} = \frac{x^2 + y^2}{10} = f(x, y) \quad \dots(*)$$

with $x_0 = 0, y_0 = 1$ & $x_1 = 0.1$ then $h = x_1 - x_0 = 0.1$

By Runge Kutta Method of fourth order

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(F')$$

$$\text{when } k_1 = hf(x_0 + y_0) = 0.1 \left[\frac{0^2 + 1^2}{10} \right]$$

$$\Rightarrow k_1 = 0.01$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \left[\frac{\left(0 + \frac{0.1}{2}\right)^2 + \left(1 + \frac{0.01}{2}\right)^2}{10} \right]$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \left[\frac{\left(0 + \frac{0.1}{2}\right)^2 + \left(1 + \frac{0.01012525}{2}\right)^2}{10} \right]$$

$$\Rightarrow k_3 = 0.01012651$$

$$\& k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 \left[\frac{(0 + 0.1)^2 + (1 + 0.01012651)^2}{10} \right]$$

$$\Rightarrow k_4 = 0.010303555$$

$$\text{then by (F)} y_1 = y(0.1) = 1 + \frac{1}{6} [0.01 + 2(0.01012525 + 0.01012651) + 0.010303555]$$

$$\Rightarrow y_1 = y(0.1) = 1.01013451$$

Q.19. Apply Runge Kutta method to find $y(0.2)$ in steps of

$$0.1 \text{ if } \frac{dy}{dx} = x + y^2, \text{ given that } y = 1 \text{ when } x = 0.$$

$$\text{Sol. : Given that } \frac{dy}{dx} = x + y^2 = f(x, y) \quad \dots(1)$$

with $x_0 = 0, y_0 = 1, h = 0.1$ & $x_n = x_0 + nh$

$$\Rightarrow x_1 = 0.1, x_2 = 0.2$$

By Runge Kutta Method

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(*)$$

$$\text{where } k_1 = hf(x_0, y_0) = (0.1) [0 + 1^2] = 0.1$$

$$\& k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.1}{2}\right)^2 \right]$$

$$k_2 = 0.11525$$

$$\therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \left[\left(0 + \frac{0.1}{2}\right) + \left(1 + \frac{0.11525}{2}\right)^2 \right]$$

$$\Rightarrow k_3 = 0.11686$$

$$\& k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) [(0 + 0.1) + (1 + 0.11686)^2]$$

$$\Rightarrow k_4 = 0.13474$$

Put these values in (*)

$$y_1 = y(0.1) = 1 + \frac{1}{6} [0.1 + 2(0.11525 + 0.11686) + 0.13474]$$

$$\Rightarrow y_1 = y(0.1) = 1.11649$$

$$\text{Now } y_2 = y(x_2) = y_1 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(**)$$

where $k_1 = hf(x_1, y_1) = (0.1) [0.1 + (1.11649)^2]$

$$\Rightarrow k_1 = 0.13465$$

$$\& k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1$$

$$\left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.11649 + \frac{0.13465}{2}\right)^2\right]$$

$$\Rightarrow k_2 = 0.15514$$

$$\text{since } k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 \left[\left(0.1 + \frac{0.1}{2}\right) + \left(1.11649 + \frac{0.15514}{2}\right)^2\right]$$

$$\Rightarrow k_3 = 0.15758$$

$$\& k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 [(0.1 + 0.1) + (1.11649 + 0.15758)^2]$$

$$\Rightarrow k_4 = 0.18233$$

Put these values in (**)

$$y_2 = y(0.2) = 1.11649 + \frac{1}{6} [0.13465 + 2(0.15514 + 0.15758) + 0.18233]$$

$$y_2 = 1.27356$$

Q.14. Use Runge Kutta method to obtain y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and y satisfies the

$$\text{equation } \frac{dy}{dx} = 3x + y^2$$

Sol : Given that

$$\frac{dy}{dx} = f(x, y) = 3x + y^2 \quad \dots(1)$$

$$x_0 = 1, y_0 = 1.2, x_1 = 1.1 \text{ then } h = x_1 - x_0 = 0.1$$

By Runge Kutta Method

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(*)$$

where

$$k_1 = hf(x_0, y_0) = 0.1[3(1) + (1.2^2)] = 0.444$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.444}{2}\right)^2\right]$$

$$k_2 = 0.5172$$

$$\& \therefore k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \left[3\left(1 + \frac{0.1}{2}\right) + \left(1.2 + \frac{0.5172}{2}\right)^2\right]$$

$$\Rightarrow k_3 = 0.5278$$

$$\& k_4 = hf(x_0 + h, y_0 + k_3) = 0.1[3(1 + 0.1) + (1.2 + 0.5278)^2]$$

$$\Rightarrow k_4 = 0.6285$$

Put these values in (*)

$$= y(1.1) = 1.2 + \frac{1}{6} [0.444 + 2(0.5172 + 0.5278) + 0.6285]$$

$$y_1 = y(1.1) = 1.727$$

Q.15. Using Runge Kutta method to fourth order to solve

$$\frac{dy}{dx} = xy \text{ for } x = 1.2 \text{ initially } x = 1, y = 2 \text{ (take } h = 0.1)$$

Sol : Given that : $\frac{dy}{dx} = f(x, y) = xy \quad \dots(1)$

$$\text{with } x_0 = 1, y_0 = 2, h = 0.1 \& \therefore x_n = x_0 + nh$$

$$\Rightarrow x_1 = 1.1, x_2 = 1.2$$

By Runge Kutta Method (fourth order)

$$y_1 = y(x_1) = y_0 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(*)$$

where

$$k_1 = hf(x_0, y_0) = (0.1) [1(2)] = 0.2$$

$$\& k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1) \left[\left(1 + \frac{0.1}{2}\right) \cdot \left(2 + \frac{0.2}{2}\right)\right]$$

$$\Rightarrow k_2 = 0.2205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1) \left[\left(1 + \frac{0.1}{2}\right) \cdot \left(2 + \frac{0.2205}{2}\right)\right]$$

$$\Rightarrow k_3 = 0.2215$$

$$\& k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) [(1 + 0.1) (2 + 0.2215)]$$

$$k_4 = 0.2443$$

Put these values in (*)

$$y_1 = y(1.1) = 2 + \frac{1}{6} [0.2 + 2(0.2205 + 0.2215) + (0.2443)]$$

$$y_1 = y(1.1) = 2.2583$$

Again by Runge Kutta Method

$$y_2 = y(x_2) = y_1 + \frac{1}{6} [k_1 + 2(k_2 + k_3) + k_4] \quad \dots(**)$$

where

$$k_1 = hf(x_1, y_1) = (0.1) [(1.1) (2.2583)] = 0.24841$$

$$\begin{aligned} & \& k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \\ & = (0.1) \left[\left(1.1 + \frac{0.1}{2}\right) \left(2.2583 + \frac{0.2454}{2}\right) \right] \\ \Rightarrow & k_2 = 0.27399 \end{aligned}$$

$$\begin{aligned} & \& k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) \\ & = (0.1) \left[\left(1.1 + \frac{0.1}{2}\right) \left(2.2583 + \frac{0.27399}{2}\right) \right] \\ \Rightarrow & k_3 = 0.27546 \end{aligned}$$

$$\begin{aligned} & \& k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) [(1.1 + 0.1)(2.2583 + 0.27546)] \\ \Rightarrow & k_4 = 0.304050 \end{aligned}$$

Put these values in (**)

$$y_2 = y(1.2) = 2.2583 + \frac{1}{6} [0.24841 + 2(0.27399) + 0.27546 + 0.30405]$$

$$\Rightarrow y_2 = y(1.2) = 2.53353$$

Predictor and Corrector Method [Milne's Method]

Given that $y' = \frac{dy}{dx} = f(x, y)$

x_0, x_1, x_2, x_3 & y_0, y_1, y_2, y_3 & h then $y(x_i)$
i.e.

(1) Milne's Predictor (Extrapolation) formula

$$y_4 = y_0 + \frac{4}{3} h [2y'_1 - y'_2 + 2y'_3]$$

(2) Milne's Corrector Formula

$$y_1^{(2)} = y_2 + \frac{h}{3} [y'_1 + 4y'_2 + 2y'_3]$$

$$\& y_1^{(2)} = y_2 + \frac{h}{3} [y'_1 + 4y'_2 + 2y'_3] \text{ and so on}$$

Q.16. Find $y(2)$ if $y(x)$ is a solution of $\frac{dy}{dx} = \frac{x+y}{2}$ assuming

$$y(0) = 2, y(0.5) = 2.636, y(1.0) = 3.595 \text{ \& } y(1.5) = 4.968$$

Sol.: Given that

$$y' = \frac{dy}{dx} = \frac{x+y}{2} = f(x, y)$$

$$\begin{aligned} & \& x_0 = 0, x_1 = 0.5, x_2 = 1.0, x_3 = 1.5 \text{ \& } x_4 = 2.0 \\ & y_0 = 1, y_1 = 2.636, y_2 = 3.595, y_3 = 4.968. \\ & y_4 = ?, h = 0.5 \end{aligned}$$

$$y' = \frac{dy}{dx} = \frac{x+y}{2} = f(x, y)$$

$$y_4 = ?, h = 0.5$$

By Milne's Predictor Method

$$y_4 = y_0 + \frac{4}{3} h [2y'_1 - y'_2 + 2y'_3] \quad \dots(A)$$

By (1)

$$y'_1 = f(x_1, y_1) = \frac{0.5 + 2.636}{2} = 1.568$$

$$y'_2 = f(x_2, y_2) = \frac{1.0 + 3.595}{2} = \frac{4.595}{2} = 2.2975$$

$$y'_3 = f(x_3, y_3) = \frac{1.5 + 4.968}{2} = 3.234$$

Put these values in eq. (A)

$$y_4 = 2 + \frac{4(0.5)}{3} [2(1.568) - 2.2975 + 2(3.234)]$$

$$y_4 = y(2.0) = 6.8710$$

Now by corrector formula

$$y_4^{(2)} = y_2 + \frac{h}{3} [y'_1 + 4y'_2 + y'_3] \quad \dots(B_1)$$

By (1)

$$y'_4 = f(x_4, y_4) = \frac{2 + 6.8710}{2} = 4.4355$$

By (1)

$$y_4^{(2)} = 3.595 + \frac{0.5}{2} [2.2975 + 4(3.234) + 4.4355]$$

$$y_4^{(2)} = 6.87316$$

Again by corrector formula

$$y_4^{(3)} = 2 + \frac{h}{3} [y'_1 + 4y'_2 + y_4^{(2)}] \quad \dots(B_2)$$

$$\text{By (1)} \quad y_4^{(3)} = f(x_4, y_4^{(2)}) = \frac{2 + 6.87316}{2} = 4.43658$$

Put in (B₁)

$$y_1^{(2)} = 3.595 + \frac{0.5}{3} [2.2975 + 4(3.234) + 4.43658]$$

$$y_1^{(2)} = 6.87335$$

$$y_1^{(2)} = y_1^{(1)}$$

$$y_1 = y(2.0) = 6.873$$

Ans.

Objectives

- Using Euler's method, the value of $y(0.2)$ for $\frac{dy}{dx} = y^2 + x^2$, $y(0) = 1$, $h = 0.2$ is :
 (a) 1.2 (b) 1.02
 (c) 1.21 (d) None of these
- Using Euler's method, the value of $y(0.2)$ for $\frac{dy}{dx} = 1 - y$ with $y(0) = 0$, $h = 0.1$ is :
 (a) 0.1 (b) 0.11
 (c) 0.19 (d) 0.20
- The value of $y(0.4)$ using Euler's method from $\frac{dy}{dx} = 1 - 2xy$ with $y(0) = 0$, $h = 0.2$ is :
 (a) 0.3 (b) 0.384
 (c) 0.33 (d) None of them
- Using Euler's method for the solution $\frac{dy}{dx} = y^2 - x^2$ with $y(0) = 1$ in steps of $h = 0.1$ the third approximate solution is :
 (a) 1.2348 (b) 1.3348
 (c) 1.3648 (d) 0.2348
- Using Runge-Kutta four order method for solving $\frac{dy}{dx} = x + y$ with $y(0) = 1$, the values of $y(0.1)$ is :
 (a) 1.0103 (b) 1.003
 (c) 1.1103 (d) None of these
- Using Runge-Kutta method for solving $\frac{dy}{dx} = \frac{1}{x+y}$ with $y(0) = 1$, $h = 0.5$ the value of K_3 is :

- (a) 0.5 (b) 0.3333
(c) 0.3 (d) 0.35294
- Using Runge Kutta method to solve $10 \frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, $h = 0.1$ the value of K_1 is :
 (a) 0.1 (b) 0.01
 (c) 0.001 (d) None of these
- The numerical technique of solving the ordinary differential equation is used when :
 (a) Differential equation can be solved by analytic method
 (b) Some differential equation cannot be solved by analytic method
 (c) Differential equation of higher order
 (d) None of them
- Milne's predictor formula for the solution of $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ is :
 (a) $y_4 = y_0 + \frac{h}{3} [2f_1 - f_2 + 2f_3]$ (b) $y_4 = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3]$
 (c) $y_4 = y_0 + \frac{h}{2} [2f_1 - f_2 + 2f_3]$ (d) None of these
- Milne's corrector formula for the solution of $\frac{dy}{dx} = f(x, y)$ when $y(x_0) = y_0$:
 (a) $y_4 = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4]$ (b) $y_4 = y_0 + \frac{h}{3} [f_2 + 4f_3 + f_4]$
 (c) $y_4 = y_2 + \frac{h}{3} [f_2 - 4f_3 + f_4]$ (d) None of these
- By Runge Kutta of fourth order the value of y at the end of first sub interval for the equation $\frac{dy}{dx} = f(x, y)$ with :
 (a) $y_1 = y_0 + \frac{1}{6} [k_1 + 4k_2 + k_3]$
 (b) $y_1 = y_0 + \frac{1}{6} [2k_1 + 2k_2 + k_3 + k_4]$

$$(c) y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$(d) y_1 = y_0 + \frac{1}{6} [k_1 + k_2 + 2k_3 + 2k_4]$$

12. By Euler's method n^{th} approximate value of y at $x = x_n = x_0 + nh$ is :

$$(a) y_n = y_0 + hf(x_{n-1}, y_{n-1}) \quad (b) y_n = y_0 - hf(x_{n-1}, y_{n-1})$$

$$(c) y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}) \quad (d) y_n = y_{n-1} + hf(x_n, y_n)$$

13. By Runge-Kutta method of fourth order for $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ & h , the value of k_4 is :

$$(a) hf(x_0, y_0) \quad (b) 4hf(x_0, y_0)$$

$$(c) hf(x_0 + h, y_0 + k_3) \quad (d) 4hf(x_0 + h, y_0 + k_3)$$

Answer

- | | | | | |
|--------|--------|--------|-------|--------|
| 1.(c) | 2.(b) | 3.(c) | 4.(c) | 5.(c) |
| 6.(d) | 7.(b) | 8.(b) | 9.(b) | 10.(a) |
| 11.(c) | 12.(c) | 13.(d) | | |

□□□

आपकी सफलता का राज़

यशराज

• बी.ए., बी.कॉम., बी.एस-सी. •
बी.बी.ए. एवं एम.ए., एम.कॉम., एल.
एल.बी., एवं बी. एड
की परीक्षाओं के लिए सभी विषयों पर
सरल अध्ययन के रूप में प्रकाशित
एक मात्र सर्वश्रेष्ठ पुस्तक

YASHRAJ

की पुस्तकें पढ़िये
सफलता की ओर बढ़िये

हिन्दी एवं English माध्यम में प्रकाशित