YASHRAJ

Elements of Discrete Mathematics

[For B.Sc., B.A. III[rd] year
Maths Group A, Paper-II]

**A Unique Book
According to New Education Policy**

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Indore

New Edition

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Ph. : (0731) 2453532

Rs. 80/-

NEW SYLLABUS

B.Sc., B.A. : Third year

ELEMENTS OF DISCRETE MATHEMATICS

[Mathematics : Group A, Paper-II]

Course Code- S3-MATH2D

Unit-I

Indian logic- Origins, The schools Vaisheshika, Catuskoti, Nyaya, Jain logic, Buddhist logic, Navya-Nyaya, Influence of Indian logic on modern logic, Boolean Logic and Indian Thoughts.

Relations- Bianry, Inverse, Composite and Equivalence relation, Equivalence classes and its properties, Partition of a set, Partial order relation, Partially ordered and Totally ordered sets, Hasses diagram.

Lattices- Definiton and examples, Dual, bounded, distributive and complemented lattices.

Unit-II

Boolean Algebra- Definition and properties, Switching circuits and its applications, Logic gates and circuits.

Boolean functions- Disjunctive and conjunctive normal forms, Bool's expansion theorem.

Minimize the Boolen function using Karnaugh Map.

Unit-III

Graph–Definition, Type of Graphs, Subgraphs, Walk, path, circuits, connected and disconnected graph. Euler graph. Hamilton path and circuits, Dijkstra's Algorithem for shortest paths in weighted graph.

Unit-IV

Trees and its properties, Rooted tree, Binary tree, Spanning tree, Rank and nullity of a graph. Kruskal's Algorithem and Prim's Algorithem.

Matrix representation of graphs : Incidence and Adjacency matrix. Cut set and its properties. Planar graph (definition) Kuratowski's two graphs.

Publisher :	:	**DEVI AHILYA PRAKASHAN** 26, Subhash Chowk, Indore- Ph. : 2453532
Edition	: New	Publisher : ©
Price	: 80/-	
Printers : Sanjari Printers	Type Setting : Sanjari Computers	

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आपकी सफलता का राज़

यशराज

बी.ए., बी.कॉम., बी.एस-सी.
एवं एम.ए., एम.कॉम., एल. एल. बी.
एवं बी. एड
की परीक्षाओं के लिए सभी विषयों पर
सरल अध्ययन के रूप में प्रकाशित
एक मात्र सर्वश्रेष्ठ पुस्तक

YASHRAJ

की पुस्तकें पढ़िये
सफलता की ओर बढ़िये

हिन्दी एवं English माध्यम में प्रकाशित

Unit-I

Indian logic- *Origins, The schools Vaisheshika, Catuskoti, Nyaya, Jain logic, Buddhist logic, Navya-Nyaya, Influence of Indian logic on modern logic, Boolean Logic and Indian Thoughts.*

Relations- *Bianry, Inverse, Composite and Equivalence relation, Equivalence classes and its properties, Partition of a set, Partial order relation, Partially ordered and Totally ordered sets, Hasses diagram.*

Lattices- *Definiton and examples, Dual, bounded, distributive and complemented lattices.*

Relation

Let A and B be two non-empty sets, then the relation from A to B is a subset of $A \times B$ and denoted as R, it is define as

$R = \{(a, b) : a \in A \text{ and } b \in B\}$

i.e. $(a, b) \in R$ or aRb.

Note– $R \subseteq A \times B$. i.e. R is subset of $A \times B$.

Types of Relation

1. **Reflexive Relation :** A relation $R \subseteq A \times A$ is reflexive if
   $$\forall\, a \in A \Rightarrow aRa \text{ or } (a, a) \in R$$
2. **Symmetric Relation :** A relation $R \subseteq A \times A$ is symmetric if
   $$aRb \Rightarrow bRa \qquad \text{for } a, b \in A$$
3. **Anti–Symmetric :** A relation $R \subseteq A \times A$ is anti–symmetric if
   $$aRb \text{ and } bRa \Rightarrow a = b \qquad \text{for } a, b \in A$$
4. **Transitive :** A relation $R \subseteq A \times A$ is transitive if
   $$aRb \text{ and } bRc \Rightarrow a\,R\,c \qquad \text{for } a, b, c \in A$$
5. **Identity :** A relation $R \subseteq A \times A$ is identity relation if
   $$aRb \subseteq a = b \qquad A\forall\, a, b \in A$$
6. **Inverse Relation :** Let R be a relation from A to B i.e. $R \subseteq A \times B$ then the inverse relation of R is denoted as R^{-1} and define as
   $R^{-1} = \{(y, x) : (x, y) \in R, x \in A \text{ and } y \in B\}$
   i.e. $R^{-1} \subseteq B \times A$
   i.e. $xRy \Rightarrow yR^{-1}x$

Composite–Relation : Let A, B and C be three non empty sets, let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations then composite relation of R and S is a relation from A to C i.e. $SOR \subseteq A \times C$ and define as

SOR = {(x, z) : ∃ y ε B such that (x, y) ε R and (y, z) ε S where x ε A and zε C}

i.e. (x, y) ε R, (y, z) ε S ⇒ (x, z) ε SOR

Equivalence–Relation

A relation R ⊆ A × A is equivalence relation if :

(1) R is reflexive

(2) R is symmetric

and (3) R is transitive. ■

Q.1. Let I be set of integers and the relation xRy ⇔ x – y is an even integer, then prove that R is an equivalence relation, where x, y ε I.

Sol. Let I = [0, ± 1, ± 2, ± 3, ...} be set of all integers and relation R define as

aRb ⇔ a – b is an even integer

⇔ a – b is divisible by 2. ...(*)

then we have to show that R is an equivalence relation, i.e. we have to show that

1. R is reflexive : Let a ε I.

∴ a – a = 0

and 0 is divisible by 2

⇒ aRa By (*)

2. R is symmetric : Let a, b ε I such that aRb then we have to show that

bRa

∵ aRb

⇒ a – b is divisible by 2 by (*)

⇒ – (b – a) is also divisible by 2

⇒ (b – a) is also divisible by 2

⇒ b R a

⇒ R is symmetric

3. R is transitive : Let a, b, c ε I such that aRb and bRc then we have to show that aRc.

∵ aRb ⇒ a – b is divisible by 2 ...(1)

& bRc ⇒ b – c is divisible by 2 ...(2)

By (1) & (2)

(a – b) + (b – c) is divisible by 2

⇒ a – c is divisible by 2

⇒ aRc

⇒ R is transitive

∵ R is reflexive, symmetric and transitive

⇒ R is an equivalence relation. ■

Q.2. Show that the relation R = {(x, y) ; x – y is divisible by 5, where x, y ε I} is an equivalence relation.

or

Show that the relation "is divisible by 5", where x, y, ε I is an equivalence relation.

Sol. Let I = {0, ± 1, ± 2, ± 3 ...} be set of integers and R = {(x, y) : x – y is divisible by 5, x, y ε I}

or xRy ⇔ x – y is divisible by 5 ...(*)

Then we have to show that R is an equivalence relation i.e. we have to show that

1. R is reflexive : Let x ε I

∵ x – x = 0

& 0 is divisible by 5

⇒ xRx ...(*)

⇒ R is reflexive

2. R is Symmetric : Let x, y ε I such that xRy then we have to show that yRx.

∵ xRy ⇒x – y is divisible by 5

⇒– (y – x) is divisible by 5

⇒y – x is divisible by 5

⇒ yRx

⇒R is symmetric

3. R is transitive : Let x, y, z ε I such that xRy and yRz then we have to show that xRz.

∵ xRy ⇒ x – y is divisible by 5 ...(1)

& yRz ⇒ y – z is divisible by 5 ...(2)

By (1) & (2)

(x – y) + (y – z) is divisible by 5

⇒ x – z is divisible by 5

⇒ xRz

⇒ R is transitive

∵ R is reflexive, symmetric and transitive

⇒ R is an equivalence relation. ■

Q.3. Prove that the relation "is equal to" in the set of all real numbers in an equivalence relation.

Sol. Let R be set of all real number and relation R define as

aRb ⇔ a = b ...(*)

then we have to show that R is an equivalence relation i.e., we have to show that :

1. R is reflexive : Let $a \varepsilon R$

$\because$ $a = a$

$\Rightarrow$ aRa

$\Rightarrow$ R is reflexive

2. R is symmetric : Let $a, b \varepsilon R$ such that aRb then we have to show that bRa

$\because$ a Rb

$\Rightarrow$ $a = b$ by (*)

or $b = a$

$\Rightarrow$ b R a by (*)

$\Rightarrow$ R is symmetric

3. R is transitive : Let $a, b, c \varepsilon R$ such that aRb and bRc then we have to show that aRc.

$\because$ $aRb \Rightarrow a = b$ ...(1)

& $bRc \Rightarrow b = c$ ...(2)

By (1) & (2) $a = c$

$\Rightarrow$ aRc by (*)

$\Rightarrow$ R is transitive

$\because$ R is reflexive, symmetric and transitive

$\Rightarrow$ R is an equivalence relation. ■

Q.4. If I is set of non zero integers and a relation R defined by $xRy \Leftrightarrow x^y = y^x$, where $x, y \varepsilon I$ then prove that R is an equivalence relation.

Sol. Let $I = \{\pm 1, \pm 2, \pm 3 ...\}$ be set of non zero integers and the relation R defined as

$$xRy \Leftrightarrow x^y = y^x \quad ...(*)$$

then we have to show that R is an equivalence relation i.e. we have to show that.

1. R is reflexive : Let $x \varepsilon I$

$\because$ $x^x = x^x$

$\Rightarrow$ xRx By (*)

$\Rightarrow$ R is reflexive.

2. R is symmetric : Let $x, y \varepsilon I$ such that xRy then we have to show that yRx

$\because$ xRy

$\Rightarrow$ $x^y = y^x$ By (*)

or $y^x = x^y$

$\Rightarrow$ yRx By (*)

$\Rightarrow$ R is symmetric.

3. R is transitive : Let $x, y, z \varepsilon I$ such that xRy and yRz then we have to show that xRz.

$\because$ xRy

$\Rightarrow$ $x^y = y^x$ ...(1)

and yRz

$\Rightarrow$ $y^z = z^y$ ...(2)

By (1) $x^y = y^x$

$(x^y)^z = (y^x)^z$

$(x^z)^y = (y^z)^x$

$(x^z)^y = (z^y)^x$ By (2)

$(x^z)^y = (z^x)^y$

$x^z = z^x$

$\Rightarrow$ xRz By (*)

$\Rightarrow$ R is transitive.

Hence R is an equivalence relation. ■

Q.5. Let N be set of natural numbers and the relation R be defined in N × N by

$$(a, b) R (c, d) \Leftrightarrow a + d = b + c$$

where $a, b, c, d \varepsilon N$, then prove that R is an equivalence relation.

Sol. Let N be set of natural number and

$$N \times N = \{(a, b) : a \varepsilon N, b \varepsilon N\}$$

and relation R defined by

$$(a, b) R (c, d) \Leftrightarrow a + d = b + c \quad ...(*)$$

Then we have to show that R is an equivalence relation i.e. we have to show that

1. R is reflexive : Let $(a, b) \varepsilon N \times N$

$\because$ $a + b = b + a$

$\Rightarrow$ (a, b) R (a, b) by (*)

$\Rightarrow$ R is reflexive

2. R is symmetric : Let (a, b), (c, d) ε N × N such that (a, b)

R (c, d) then we have to show that (c, d) R (a, b)

$\because$ (a, b) R (c, d)

$\Rightarrow$ $a + d = b + c$ By (*)

or $b + c = a + d$

$\Rightarrow$ $c + b = d + a$

$\Rightarrow$ (c, d) R (a, b) By (*)

$\Rightarrow$ R is symmetric

3. R is transitive : Let (a, b), (c, d) and (e, f) ε N × N such that (a, b) R (c, d) and (c, d) R (e, f) then we have the show that (a, b) R (e, f)

$\because$ (a, b) R (c, d)

$\Rightarrow$ $a + d = b + c$ ...(1)

& $\because$ (c, d) R (e, f)

$\Rightarrow$ $c + f = d + e$ ...(2)

By (1) + (2)

$$a + d + c + f = b + c + d + e$$

$$a + f = b + e$$

$\Rightarrow$ (a, b) R (e, f)

$\Rightarrow$ R is transitive

$\because$ R is reflexive, symmetric and transitive

$\Rightarrow$ R is equivalence relation ■

Q.6. Prove that the relation R on the set I of all integers defined by

(a, b) ε R $\Leftrightarrow$ a – b is divisible by n, $1 \leq n \varepsilon N$ is an equivalence relation.

Sol. Let I be set of all integers and relation R is define by

aRb $\Leftrightarrow$ a – b is divisible by n ...(*)

then we have to show that R is an equivalence relation i.e. we have to show that

1. R is reflexive : Let a ε I

$\because$ $a - a = 0$

and 0 is divisible by n

$\Rightarrow$ aRa by (*)

$\Rightarrow$ R is reflexive

2. R is symmetric : Let a, b ε I such that aRb then we have to show that bRa

$\because$ aRb

$\Rightarrow$ a – b is divisible by n by (*)

$\Rightarrow$ – (b – a) is divisible by n

$\Rightarrow$ b – a is also divisible by n

$\Rightarrow$ bRa By (*)

$\Rightarrow$ R is symmetric

3. R is transitive : Let a, b, c ε I such that aRb and bRc then we have to show that aRc

$\because$ aRb

$\Rightarrow$ a – b is divisible by n ...(1) by (*)

& $\because$ bRc

$\Rightarrow$ b – c is divisible by n ...(2)

By (1) & (2)

(a – b) + (b – c) is divisible by n

$\Rightarrow$ a – c is divisible by n

$\Rightarrow$ aRc

$\Rightarrow$ R is transitive

$\because$ R is reflexive, symmetric and transitive

$\Rightarrow$ R is transitive. ■

Q.7. If R is an equivalnce relation on a set A then prove that R^{-1} is also an equivalence relation on A.

Sol. Let R be an equivalence relation on a set A and R^{-1} be inverse relation of R on A.

Let a, b ε A then

$aRb \Rightarrow bR^{-1}a$ ...(*)

We have to show that R^{-1} is an equivalence relation i.e. we have to show that

1. R^{-1} is reflexive : Let a ε A

$\because$ R is reflexive

$\Rightarrow$ aRa

$\Rightarrow$ $aR^{-1}a$ by (*)

$\Rightarrow$ R^{-1} is reflexive

2. R^{-1} is symmetric : Let a, b ε A

$\because$ R is symmetric

$\Rightarrow$ $aRb \Rightarrow bRa$

$\because$ $aRb \Rightarrow bR^{-1}a$ ...(1) by (*)

& $bRa \Rightarrow aR^{-1}b$ ...(2) by (*)

By (1) & (2)

$bR^{-1}a \Rightarrow aR^{-1}b$

$\Rightarrow R^{-1}$ is symmetric

3. R^{-1} is transitive : Let a, b, c ε A.

∵ R is transitive

⇒ aRb and bRc ⇒ aRc

∵ $aRb \Rightarrow bR^{-1}a$ ...(1) by (*)

$bRc \Rightarrow cR^{-1}b$ ...(2) by (*)

& $aRc \Rightarrow cR^{-1}a$ ...(3) by (*)

By (2), (1) & (3)

$cR^{-1}b$ and $bR^{-1}a \Rightarrow cR^{-1}a$

$\Rightarrow R^{-1}$ is transitive

∵ R^{-1} is reflexive, symmetric and transitive

⇒ R^{-1} is an equivalence relation. ■

Q.8. If R and S be equivalence relations in the set X, then prove that R ∩ S is an equivalence relation in X.

Sol. Let R and S be two equivalence relation in the set X, then we have to show that R ∩ S is also equivalence relation i.e. we have to show that

1. R ∩ S is reflexive : Let a ε X

∵ R and S are reflexive

⇒ (a, a) ε R and (a, a) ε S

⇒ (a, a) ε R ∩ S

⇒ R ∩ S is reflexive

2. R ∩ S is symmetric : Let a, b ε X.

∵ R and S are symmetric

⇒ (a, b) ε R ⇒ (b, a) ε R ...(1)

and (a, b) ε S ⇒ (b, a) ε S ...(2)

By (1) & (2)

(a, b) ε R ∩ S ⇒ (b, a) ε R ∩ S

⇒ R ∩ S is symmetric

3. R ∩ S is transitive : Let a, b, c ε X

∵ R and S are transitive then

(a, b) ε R and (b, c) ε R ⇒ (a, c) ε R ...(1)

and (a, b) ε S and (b, c) ε S ⇒ (a, c) ε R ...(2)

By (1) & (2)

(a, b) ε R ∩ S and (b, c) ε R ∩ S ⇒ (a, c) ε R ∩ S

⇒ R ∩ S is transitive

∵ R ∩ S is reflexive, symmetric and transitive

⇒ R ∩ S is an equivalence relation ■

Q.9. If R^{-1} and S^{-1} are inverse relations of the relation R and S respectively, then prove that

$$(SOR)^{-1} = R^{-1} O S^{-1}$$

Sol. Let A, B and C be three non empty sets and R be relation from A to B i.e. $R \subseteq A \times B$ and S be relation from B to c i.e. $S \subseteq B \times C$, then by definition of composition relation SOR is relation from A to C i.e. $SOR \subseteq A \times C$ Now by definition of inverse relation.

$(SOR)^{-1} \subseteq C \times A$ is inverse relation of SOR.

Let $(z, x) \varepsilon (SOR)^{-1}$ be arbitrary, for $z \varepsilon C$, $x \varepsilon A$

∵ $(z, x) \varepsilon (SOR)^{-1}$

⇒ $(x, z) \varepsilon SOR$

⇒ ∃ y ε B such that $(x, y) \varepsilon R$ and $(y, z) \varepsilon S$

⇒ $(y, x) \varepsilon R^{-1}$ and $(z, y) \varepsilon S^{-1}$

∵ $(z, y) \varepsilon S^{-1}$ and $(y, x) \varepsilon R^{-1}$

⇒ $(z, x) \varepsilon R^{-1}OS^{-1}$

∴ for $(z, x) \varepsilon (SOR)^{-1}$

⇒ $(z, x) \varepsilon R^{-1} O S^{-1}$

∵ (z, x) is arbitrary

⇒ $(SOR)^{-1} \subseteq R^{-1} O S^{-1}$ ...(1)

Similarly we can prove that

$R^{-1} O S^{-1} \subseteq (SOR)^{-1}$ ...(2)

By (1) & (2)

$(SOR)^{-1} = R^{-1} O S^{-1}$ ■

Q.10. Give an example of a relation which is reflexive but neither symmetric nor transitive.

Sol. Let A = {a, b, c} be any set and

R = {(a, a), (a, b), (b, b), (b, c), (c, c)} is a relation in A.

1. Reflexive ∵ a, b, c ε R and

aRa, bRb and cRc

i.e. every element of A is related to itself

$\Rightarrow$ R is reflexive

2. Symmetric $\because$ aRb but b $\not R$ a i.e. (b, a) $\notin$ R

$\Rightarrow$ R is not symmetric

3. Transitive : Here (a, b) ε R i.e. aRb

and (b, c) ε R $\Rightarrow$ bRc

but (a, c) $\notin$ R i.e. a $\not R$ c

$\Rightarrow$ R is not a transitive relation

Partition of a set :

Let X be a non empty set and P be a set ot subset of X i.e. P = $\{A_1, A_2, A_3, ...\}$ then P is called partition of X if

1. $A_1 \cup A_2 \cup A_3 \cup .... = X$

and 2. If $A_i \neq A_j$ then

$A_i \cap Aj = \phi$ ■

Partial order Relation [P.O.R.]

A relation R on a set X is said to be partial order relation if it is :

(1) Reflexive : $\forall$ a ε X $\Rightarrow$ aRa

(2) Anti–symmetric : Let a, b ε X, If aRb and bRa

$\Rightarrow$ a = b

(3) Transitive : Let a, b, c and aRb and bRc

$\Rightarrow$ a R c

Comparable–Elements [Law of Dichotomy]

Let x, y ε A be two elements and $\leq$ be a relation define in A then elements x and y are said to be comparable element if either $x \leq y$ or $y \leq x$.

Partially ordered set [Poset]

A set A with a partial ordering relation $\leq$ on A is called a partially ordered set (POSet) and it is denoted as (A, $\leq$).

Total ordering Relation [Total order relation]

A partial order relation defined on a set A is called total order relation if every two elements in A are comparable.

Hasse–Diagram : A diagramatically representation of a partially ordered set (POSet) is known as Hasse–diagram of the relation.

Maximal–Element : Let (A, $\leq$) be a partially ordered set

(POset) then an element u ε A is called a maximal element of A if for each x ε A such than

$u \leq x$

$\Rightarrow$ $u = x$

Note : A maximal element in set is greater than each element of the set.

Minimal Element : Let (A, $\leq$) be a partially ordered set (POset) then an element $l \varepsilon$ A is called a minimal element of A if for each x ε A such that

$x \leq l$

$\Rightarrow$ $l = x$

Note : A minimal element in set is the smaller less than each element of the set.

First–Element & Last Element

Let (A, $\leq$) be a partially ordered set (POset) then an element a ε A is called first element if

$a \leq x \quad \forall\, x \varepsilon A$

and an element b ε A is called last element if

$x \leq b \quad \forall\, x \varepsilon A$

Note : (1) First element $\equiv$ lowest element

$\equiv$ Zero element $\equiv$ lowest element

(2) Last element $\equiv$ greatest element

$\equiv$ Unit element $\equiv$ highest element

Q. 11. Let N be the set of positive integers. Prove that the relation $\leq$, where $\leq$ has its usual meaning, is a partial order relation on N.

Sol. Let N be set of all positive integers, i.e. natural number and relation `$\leq$' meaning is ``less than or equal" then we have to show that $\leq$ is partial order relation i.e. we have to show that

1. $\leq$ is reflexive : Let n ε N

$\because$ n is equal to itself

$\Rightarrow n \leq n$

$\Rightarrow \leq$ is reflexive

2. $\leq$ is Anti–symmetric :

Let m, n ε N such that $m \leq n$ and $n \leq m$ then we have to show that m = n

$\because$ $m \leq n$ ...(1)

and $n \leq m$ ...(2)

By (1) & (2) $m = n$

$\Rightarrow \leq$ is anti symmetric

3. ≤ is transitive :

Let m, n, k ε N such that m ≤ n and n ≤ k then we have to show that

$$m \le k$$

$\because$ $m \le n$ ...(1)

& $n \le k$ ...(2)

By (1) & (2) $m \le k$

$\Rightarrow$ ≤ is transitive.

Since ≤ is reflexive, anti–symmetric & transitive

$\Rightarrow$ ≤ is partial order relation. ∎

Q. 12. Let S be any class of sets. Prove that the relation of set inclusion (subset) ⊆ is a partial order relation on.

Sol. Let S be the set of sets and ⊆ means subset or equal to then we have to show that ⊆ is a partial order relation i.e.

(1) ⊆ is reflexive : Let A ε S

$\because$ A is subset of itself

i.e. A ⊆ A

$\Rightarrow$ ⊆ is reflexive

(2) ⊆ is anti–symmetric :

Let A, B ε S such that A ⊆ B and B ⊆ A then we have to show that

$$A = B$$

$\because$ $A \subseteq B$ ...(1)

& $B \subseteq A$ ...(2)

then by (1) & (2)

$$A = B$$

$\Rightarrow$ ⊆ is anti-symmetric

(3) ⊆ is transitive :

Let A, B, C ε S such that A ⊆ B and B ⊆ C then we have to show that

$$A \subseteq C$$

$\because$ $A \subseteq B$ ...(1)

and $B \subseteq C$ ...(2)

By (1) & (2)

$$A \subseteq C$$

$\Rightarrow$ ⊆ is transitive

$\because$ ⊆ is reflexive, anti–symmetric and transitive

$\Rightarrow$ ⊆ is partial order relation. ∎

Q. 13. Prove that the relation of divisibility is a partial order relation on N.

Or

Prove that the relation R = {(a, b) : aRb ⇔ a divides b, a, b ε N} is a partial order relation.

Sol. Let N be set of natural number and

R = {(a, b) : aRb ⇔ a divides b, a, b ε N} ...(1)

then we have to show that R is partial order relation, i.e. we have to show that

(1) R is reflexive : Let n ε N, $n \ge 1$

$\because$ every positive integer divide itself

$\Rightarrow$ nRn

$\Rightarrow$ R is reflexive

(2) R is anti–symmetric :

Let m, n ε N such that mRn and nRm then we have to show that m = n

$\because$ mRn

$\Rightarrow$ m divides n

$\Rightarrow \exists\ n_1 \varepsilon N$ such that

$$\frac{n}{m} = n_1$$

$\Rightarrow$ $n = n_1 m$ ...(1)

& $\because$ nRm

$\Rightarrow$ n divides m

$\Rightarrow \exists\ n_2 \varepsilon N$ such that

$$\frac{m}{n} = n_2$$

$$m = n_2 n$$

$$m = n_2(n_1 m) \quad \text{by (1)}$$

$$1 = n_1 n_2$$

It is only possible when $n_1 = n_2 = 1$

Put $n_1 = 1$ in (1)

$n = 1.m$

$m = n$

$\Rightarrow$ R is anti–symmetric

(3) R is transitive : Let m, n, k ε N such that mRn and nRk then we have to show that mRk.

$\because$ mRn

== m divides n

$\Rightarrow \exists\ n_1 \varepsilon N$ such that

$$\frac{n}{m} = n_1$$

$$n = n_1 m \quad \ldots(1)$$

$\because$ nRk

$\Rightarrow$ n divides k

$\Rightarrow \exists\ n_2 \in N$ such that

$$\frac{k}{n} = n_2$$

$$\Rightarrow \quad k = n_2 n$$

$$\Rightarrow \quad k = n_2 n_1 m \quad \text{by (1)}$$

$$\frac{k}{m} = n_2 n_1 = \text{integer}$$

$\Rightarrow$ m divides k

$\Rightarrow$ mRk

$\Rightarrow$ R is transitive

$\because$ R is reflexive, anti–symmetrix & transitive

$\Rightarrow$ R is partial order relation. ■

Q. 14 (a). Prove that a finite non empty poset (P, ≤) has at least one maximal element and at least one minimal element.

Sol. Let (P, ≤) be a finite non–empty poset.

Let x_1 be any element of P. If x_1 is not maximal element, then there exist $x_2 \in P$ such that $x_1 < x_2$.

If x_2 is not maximal, there exist $x_3 \in P$ such that $x_2 < x_3$ and so on. Since P is finite, therefore this chain must end with some element $x_k \in P$. Since, there is no element x of P such that

$$x_k < x$$

$\Rightarrow x_k$ is maximal element.

Similarly, we can prove that P has at least one minimal element.

Q. 14 (b). Prove that poset has at most one zero element and at most one unit element.

Sol. Let (P, ≤) be a poset

Suppose on contrary that P has two distinct zero element say x_1 and x_2

As x_1 is zero element then

$$x_1 \le x_2 \quad \ldots(1)$$

As x_2 is zero element then

$$x_2 \le x_1 \quad \ldots(2)$$

By (1) & (2)

$$x_1 = x_2$$

$\Rightarrow$ zero element is unique.

Now, suppose on contrary, P has two distinct unit elements say y_1 and y_2.

As y_1 is unit element then

$$y_2 \le y_1 \quad \ldots(1)$$

As y_2 is unit element then

$$y_1 \le y_2 \quad \ldots(2)$$

By (1) & (2)

$$y_1 = y_2$$

$\Rightarrow$ Unit element is unique.

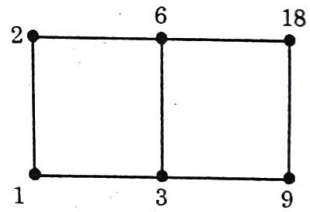
Q. 15 (a). Draw the Hasse diagram of the set of positive divisor of 18, with relation of divides.

Sol. Let A be the set of all positive divisor of 18 i.e.

A = {1, 2, 3, 6, 9, 18}

Let R be given relation i.e. aRb $\Leftrightarrow$ a divides b

$\therefore$ its Hasse–diagram is give by



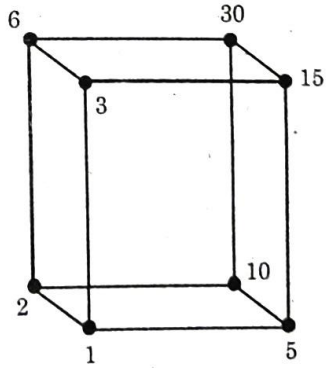
Q. 15 (b). Draw the Hasse–diagram of the set of positive divisor of 30, with relation of divides.

Sol. Let A be the set of all positive divisor of 30, i.e.

A = {1, 2, 3, 5, 6, 10, 15, 30}

Let R be given relation i.e. aRb $\Leftrightarrow$ a divides b.

$\therefore$ Its Hasse–diagram is given by



Q. 15 (c). Draw the Hasse–diagram of the set of subsets of

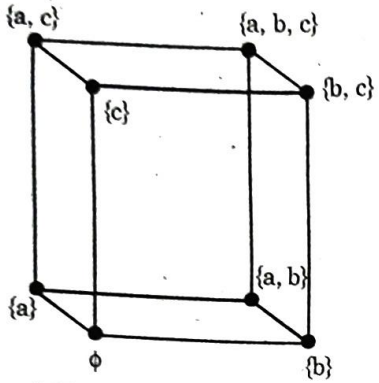
S = {a, b, c} with inclusion relation A ⊆ B.

Sol. Give that S = {a, b, c}, Let A be the set of all subsets of S i.e.

A = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}

Let R be given relation i.e. aRb ⇔ $A_1 \subseteq A_2$

Its Hasse–diagram is given below.



Q. 16. If (A, ≤) and (B, ≤) are posets, then show that (A × B, ≤) is also a poset with partial order relation define by $(a_1, b_1) \le (a_2, b_2) \Leftrightarrow a_1 < a_2$ and $b_1 < b_2$ when $a_1, a_2 \in A$, $b_1, b_2 \in B$

Sol. Let A and B be two sets then

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Given that (A, ≤) and (B, ≤) are poset, then we have to show that (A × B, ≤) is also a poset with relation

$$(a_1, b_1) \le (a_2, b_2) \Leftrightarrow a_1 \le a_2 \text{ and } b_1 < b_2 \quad ...(*)$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

To show that '≤' is partial order relation on A × B, we have to show that

(1) ≤ is reflexive.

Let $a \in A$ and $b \in B$

∵ ≤ is reflexive on A and B

⇒ $a \le a$ and $b \le b$

⇒ $(a, b) \le (a, b)$ by (*)

⇒ ≤ is reflexive on A × B.

(2) ≤ is anti–symmetric :

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that

$(a_1, b_1) \le (a_2, b_2)$ and $(a_2, b_2) \le (a_1, b_1)$

then we have to show that

$$(a_1, b_1) = (a_2, b_2)$$

∵ $(a_1, b_1) \le (a_2, b_2)$

⇒ $\begin{cases} a_1 \le a_2 & ...(1) \\ b_1 \le b_2 & ...(2) \end{cases}$ ...(1) by (*)

& $(a_2, b_2) \le (a_1, b_1)$

⇒ $\begin{cases} a_2 \le a_1 & ...(3) \\ b_2 \le b_1 & ...(4) \end{cases}$

By (1) & (3)

$a_1 = a_2$ ∵ ≤ is anti symmetric on A

& by (2) & (4)

$b_1 = b_2$ ∵ ≤ is anti symmetric on B

⇒ $(a_1, a_2) = (b_1, b_2)$

⇒ ≤ is anti–symmetric on A × B.

(3) ≤ is transitive :

Let $(a_1, b_1), (a_2, b_2)$ and $(a_3, b_3) \in A \times B$ such that $(a_1, b_1) \le (a_2, b_2)$ and $(a_2, b_2) \le (a_3, b_3)$ then we have to show that $(a_1, b_1) \le (a_3, b_3)$

∵ $(a_1, b_1) \le (a_2, b_2)$

⇒ $\begin{cases} a_1 \le a_2 & ...(1) \\ b_1 \le b_2 & ...(2) \end{cases}$ by (*)

& $(a_2, b_2) \le (a_3, b_3)$

⇒ $\begin{cases} a_2 \le a_3 & ...(3) \\ b_2 \le b_3 & ...(4) \end{cases}$ by (*)

By (1) & (3)

$a_1 \le a_2$ and $a_2 \le a_3$

⇒ $a_1 \le a_3$ ...(5) ∵ ≤ is transitive on A

& By (2) and (4)

$b_1 \le b_2$ and $b_2 \le b_3$

⇒ $b_1 \le b_3$ ...(6) ∵ ≤ is transitive on B

By (5) & (6)

$(a_1, b_1) \le (a_3, b_3)$ by (*)

⇒ ≤ is transitive on A × B.

∵ ≤ is reflexive, anti–symmetric and transitive

⇒ ≤ is partial order relation on A × B

⇒ (A × B, ≤) is a poset. ■

Q. 17. If Z be a set of integers and the relation R defined by

aRb ⇔ $a = b^r$, r ∈ N ≡ set of positive integers then prove that (Z, R) is a poset.

Sol. Let Z be set of integers and N be set of positive integers and relation R defined by

$$aRb \Leftrightarrow a = b^r, r \in N \quad ...(*)$$
$$a, b \in Z$$

then we have to show that (Z, R) is a poset, i.e. we have to show that relation R is partial order relation on Z i.e.

(1) R is reflexive :

Let $x \varepsilon Z$, we know that

$x = x^1$ $\because 1 \varepsilon N$

$\Rightarrow xRx$ By (*)

$\Rightarrow$ R is reflexive on Z

(2) R is anit-symmetric :

Let $a, b \varepsilon Z$ and $r, s \varepsilon N$ such that

aRb and bRa then we have to show that a = b.

$\because$ $aRb \Rightarrow \exists\ r \varepsilon N$ such that

$\Rightarrow$ $a = b^r$ ...(1) by (*)

& $bRa \Rightarrow \exists\ s \varepsilon N$ such that

$b = a^s$ ...(2)

By (1)

$a = b^r$

$= (a^s)^r$

$a = a^{rs}$

$\Rightarrow$ $1 = r.s.$

it is only possible when

$r = 1$ and $s = 1$ then by (1) & (2)

$a = b^1$ and $b = a^1$

$\Rightarrow$ $a = b.$

$\Rightarrow$ R is anti–symmetric on Z

(3) R is transitive

Let $a, b, c \varepsilon Z$ such that aRb and bRc then we have to show that aRc

$\because$ $aRb \Rightarrow \exists\ r \varepsilon N$ such that

$a = b^r$ ...(1)

$\because$ $bRc \Rightarrow \exists\ s \varepsilon N$ such that

$b = c^s$ ...(2)

By (1)

$a = b^r$

$= (c^s)^r$

$a = c^{r.s}$

$a = c^t$ say $r.s = t \varepsilon N$

$\Rightarrow$ aRc

$\Rightarrow$ R is transitive

$\because$ R is reflexive, symmetric and transitive on Z

$\Rightarrow$ R is partial order relation on Z $\Rightarrow$ (Z, R) is poset.

Join and Meet :

The operation V is called join and operation Λ is called meet in a poset $(p, \leq)$.

LATTICE : A partially ordered set $(A, \leq)$ is called lattice if A is closed under V and Λ, i.e. $\forall\ a, b \varepsilon A$

$\Rightarrow$ $aVb \varepsilon A$

and $a \Lambda b \varepsilon A$.

It is denoted as $(A, \leq, V, \Lambda)$

Dual–lattice : Let $(A, \leq, V, \Lambda)$ be a lattice, then the dual lattice of $(A, \leq, V, \Lambda)$ is $(A, \geq, \Lambda, V)$.

Sub–Lattice : Let $(A, \leq, V, \Lambda)$ be a lattice and $S \subseteq A$ the S is called sublattice.

if $(S, \leq, V, \Lambda)$ is a lattice.

Bounded lattice : A lattice A is called bounded lattice if its upper bound and lower bound belong to A.

Complement of an element of a lattice : Let $(A, \leq, V, \Lambda)$ be a bounded lattice. Let 0 and 1 be lower bound and upper bound of A respectively. Let $a \varepsilon A$ then complement of a is denoted as a' and define as

$a\ V\ a' = 1$

$a \Lambda a' = 0$

Complemented - Lattice :

A bounded lattice $(A, \leq, V, \Lambda)$ is called complemented lattice, if the complement of each element of it exists in lattice. ■

Q. 18. Prove that $(P(X), \subseteq, V, \Lambda)$ is a lattice, where P(X) is the power set of X, $\subseteq$ is inclusion relation and join and meet define by

$$A \vee B = A \cup B$$

and $$A \wedge B = A \cap B, \text{ where } A, B \varepsilon P(X).$$

Sol. Let $X = \{a, b, c\}$ be any set and P(X) be power set of X, i.e.

$P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a, c\}, \{b, c\}\}$

then we have to show that $(P(X), \subseteq, V, \Lambda)$ is a lattice. First we show that $(P(X), \subseteq)$ is poset i.e. we have to show that

(1) $\subseteq$ is reflexive : Let $A \varepsilon P(X)$, we know that any set is subset of itself i.e.

$A \subseteq A$

$\Rightarrow \subseteq$ is reflexive on P(X).

(2) $\subseteq$ is anti–symmetric : Let $A, B \varepsilon P(X)$ such that $A \subseteq B$ and $B \subseteq A$ then we have to show that

$A = B$

Since $A \subseteq B$ ...(1)

& $B \subseteq A$ ...(2)

By (1) & (2)

$A = B.$

$\Rightarrow \subseteq$ is anti–symmetric on P(X).

(3) $\subseteq$ is transitive : Let A, B, C ε P(X) such that $A \subseteq B$ and $B \subseteq C$ then we have to show that

$A \subseteq C$

Since $A \subseteq B$ ...(1)

& $B \subseteq C$ ...(2)

By (1) & (2) $A \subseteq C$

$\Rightarrow \subseteq$ is transitive on P(X)

Since $\subseteq$ is reflexive, anti -symmetric & transitive on P(X)

$\Rightarrow \subseteq$ is partial order relation on P(X)

$\Rightarrow$ (P(X), $\subseteq$) is poset.

Now, since

$A \wedge B = A \cup B$

and $A \wedge B = A \cap B$

We know that union and intersection of two subsets of a set is again a subset of the set

$\Rightarrow \forall$ A, B ε P(X)

$A \cup B \varepsilon P(X)$

$A \cap B \varepsilon P(X)$

$\Rightarrow$ P(X) is closed under V and $\wedge$

$\Rightarrow$ P(X) is a lattice. ■

Q. 19. Let N be set of natural number and $\leq$ means as usual meaning and join and meet for a, b ε N are defined as follows :

$$\mathbf{a \vee b = max\ (a, b)}$$

$$\mathbf{a \wedge b = min\ (a, b)}$$

then prove that N is a lattice.

Sol. Let N be set of natural numbers and for a, b ε N

$a \leq b \Leftrightarrow$ a is less than or equal to b ...(1)

$a \vee b = \max \{a, b\}$ ...(2)

$a \wedge b = \min \{a, b\}$ ...(3)

First we show that (N, $\leq$) is poset i.e. we have to show that

(1) $\leq$ is reflexive : Let a ε N, since any number is equal to itself then

$a \leq a$

$\Rightarrow \leq$ is reflexive on N

(2) $\leq$ is anti–symmetric : Let a, b ε N such that $a \leq b$ and $b \leq a$ then we have to show that

$a = b$

Since $a \leq b$ ...(1)

& $b \leq a$ ...(2)

By (1) & (2)

$a = b$

$\Rightarrow \leq$ is anti–symmetric on N.

(3) $\leq$ is transitive : Let a, b, c ε N such that $a \leq b$ and $b \leq a$ then we have to show that

$a \leq c$

Since $a \leq b$ ...(1)

& $b \leq c$ ...(2)

By (1) & (2)

$a \leq c$

$\Rightarrow \leq$ is transitive on N

Since $\leq$ is reflexive, anti–symmetric and transitive

$\Rightarrow \leq$ is partial order relation on N

$\Rightarrow$ (N, $\leq$) is poset.

$\because$ $a \vee b = \max \{a, b\}$

& $a \wedge b = \min [a, b\}$

for a, b εN

$a \leq b$ or $b \leq a$

If $a \leq b$ then is maximum and a is minimum

& $b \leq a$ then a is maximum & b is minimum

$\Rightarrow$ $a \vee b \varepsilon N$

$a \wedge b \varepsilon N$

$\Rightarrow$ N is closed under V, $\wedge$.

$\Rightarrow$ N is a lattice. ■

Q. 20. Let L be the set of all factors/divisors of 12 and Let "|" be the divisibility relation on L and

$$\mathbf{a \vee b = sup\ \{a, b\} = l.c.m.\ \{a, b\}}$$

and $\mathbf{a \wedge b = inf\ \{a, b\} = g.c.d.\ \{a, b\}}$

Prove that L is a lattice.

Sol. Let L be the set of all factors or positive divisors of 12 i.e.

$L = \{1, 2, 3, 4, 6, 12\}$

and for a, b ε L

$$a|b \Leftrightarrow a \text{ divides } b \qquad ...(*)$$

To show that L is lattice, first we show that (L, |) is poset i.e. we have to show that

(1) | is reflexive : Let a ε L since every positive integer is divisor of it self

⇒ a|a

⇒ | is reflexive

(2) | is anti–symmetric : Let a, b ε L such that aIb and bIa then we have to show that

$$a = b$$

Since a|b

⇒ a is divisor of b

⇒ ∃ n_1 ε N such that

$$\frac{b}{a} = n_1$$

$$b = n_1 a \qquad ...(1)$$

Since b|a

⇒ b is divisor of a

⇒ ∃ n_2 ε N such that

$$\frac{a}{b} = n_2$$

$$a = bn_2$$

$$a = (n_1 a)n_2 \qquad \text{by (1)}$$

$$a = n_1 n_2 a$$

$$1 = n_1 n_2$$

it is only possible when $n_1 = 1 = n_2$

Put $n_1 = 1$ in (1)

$$b = a$$

or $$a = b$$

⇒ | is anti symmetric

(3) | is transitive : Let a, b, c ε L such that a|b and b|c then we have to show that a|c Since aIb

⇒ a divides b

⇒ ∃ n_1 ε N such that

$$\frac{b}{a} = n_1$$

$$b = n_1 a \qquad ...(1)$$

& b|c

⇒ b divides c

⇒ ∃ n_2 ε N such that

$$\frac{c}{a} = n_2$$

$$c = bn_2$$

$$c = (n_1 a)n_2 \qquad \text{by (1)}$$

$$c = n_1 n_2 a$$

or $$\frac{c}{a} = n_1 n_2 = n \varepsilon N$$

⇒ a divides c

⇒ a|c

⇒ | is transitive

Since | is reflexive, anti–symmetric and transitive

⇒ (L, |) is poset.

Since $a \vee b$ = l.c.m {a, b}

$a \wedge b$ = g.c.d {a, b}

V	1	2	3	4	6	12
1	1	2	3	4	6	12
2	2	2	6	4	6	12
3	3	6	3	12	6	12
4	4	4	12	4	12	12
6	6	6	6	6	12	6
12	12	12	12	12	12	12

T_1

Λ	1	2	3	4	6	12
1	1	1	1	1	1	1
2	1	2	1	2	2	2
3	1	1	3	1	3	3
4	1	2	1	1	2	4
6	1	2	3	2	6	6
12	1	2	3	4	6	12

T_2

Since all entries of T_1 and T_2 are elements of L then L is closed under V and Λ

⇒ L is a lattice ■

Principle of Duality

The dual of any theorem of lattice is also a theorem.

Q. 21. Prove that dual of a lattice is also a lattice.

Sol. Let L be a non empty set and R be a partial order relation on non empty set L.

$\Rightarrow$ (L, R) is a poset.

Let R^{-1} be the inverse relation of R on L

$\Rightarrow$ R^{-1} is also partial order relation of L

$\Rightarrow$ (L, R^{-1}) is a poset.

Given that L is lattice for a, b ε L

$$a \vee b = \sup \{a, b\} = \text{l.u.b } \{a, b\}$$
$$a \wedge b = \inf \{a, b\} = \text{g.l.b } \{a, b\}$$

exists in L

$$\Rightarrow \quad a R (a \vee b)$$
$$\& \quad b R (a \vee b)$$
$$\Rightarrow \quad (a \vee b) R^{-1} a, \; (a \vee b) R^{-1} b \text{ by def. of } R^{-1}$$

$\Rightarrow$ aVb is lower bound of a and b with respect to R^{-1}

Now if c is any other lower bound of a and b with respect to R^{-1} then

$$c R^{-1} a \text{ and } c R^{-1} b$$
$$\Rightarrow \quad a R c \text{ and } b R c$$

$\Rightarrow$ c is the upper bound of a and b with respect to R.

$\Rightarrow$ $(a \vee b) R c$ [$\because$ $a \vee b = \sup \{a, b\}$

$\Rightarrow$ $c R_1 (a \vee b)$

$\Rightarrow$ (aVb) is the g.l.b. of a and b with respect to R^{-1}

$\Rightarrow$ $a \vee b = \inf \{a, b\}$ in (L, R^{-1})

Similarly, we can prove that

$a \wedge b = \text{sub } \{a, b\}$ in (L, R^{-1}) ■

Q. 22. Let 1 and 0 be the upper bound and lower bound for the lattice (L, ≤), then for each a ε L, prove that

(1) $a \vee 1 = 1$ and $a \wedge 1 = a$
(2) $a \vee 0 = a$ and $a \wedge 0 = 0$ **(*)**

Sol. Let (L, ≤) be a lattice, then for a, b ε L

$$a \vee b = \sup \{a, b\} = \text{l.u.b } \{a, b\}$$
$$a \wedge b = \inf \{a, b\} = \text{g.l.b } \{a, b\}$$

Let 1 be upper bound and O be lower bound of L.

(1) $\because$ 1 is the upper bound of L, then for a ε L

$$a \vee 1 \le 1 \quad \ldots(1)$$

Since aVl is supremum of a and l by (*)

$$\Rightarrow \quad 1 \le a \vee 1$$

By (1) & (2) ...(2)

$$a \vee 1 = 1$$

Now, by (*) $a \wedge l$ is infimum of a and 1

$$\Rightarrow \quad a \wedge 1 \le a$$

Since $a \le a$ ...(3)

&

$$a \le 1$$
$$\Rightarrow \quad a \wedge a \le a \wedge 1$$
$$a \le a \wedge 1 \quad \ldots(4) \text{ by Idempotent (a)}$$

By (3) & (4)

$$a \wedge 1 = a$$

(2) Given that 0 is lower bound of L then for a ε L

$$0 \le a$$

and

$$a \le a$$
$$\Rightarrow \quad 0 \vee a \le a \vee a$$
$$0 \vee a \le a \quad \ldots(5) \text{ by Idempotent}$$

Now by (*) 0Va is supremum of 0 and a

$$\Rightarrow \quad a \le 0 \vee a$$

By (5) & (6) ...(6)

$$0 \vee a = a$$

Now, by (*) $a \wedge a$ is infimum of 0 and a

$$a \wedge a \le 0 \quad \ldots(7)$$
$$\because \quad 0 \le a$$

and

$$0 \le 0$$
$$\Rightarrow \quad 0 \wedge 0 \le a \wedge 0$$
$$0 \le a \wedge 0 \quad \ldots(8)$$

By (7) & (8)

$$a \wedge 0 = 0$$ ■

Q. 23. Prove that for any positive integers a and b
ℓ.c.m {a, g.c.d. {a, b}} = a
g.c.d. {a, l.c.m. {a, b}} = a.

Sol. Let N be set of all positive integers and '≤' defined by

$$a \le b \Leftrightarrow a \text{ divides } b$$

then (N, ≤) be a lattice, with

$$a \vee b = \text{l.c.m. } \{a, b\} \quad \ldots(1)$$

& $a \wedge b = \text{g.c.d. } \{a, b\}$ ...(2)

Now, by as absorption law

$$a \wedge (a \vee b) = a \quad \ldots(3)$$
$$a \vee (a \wedge b) = a \quad \ldots(4)$$

Now l.c.m. {a, g.c.d. {a, b}}

$$= \text{l.c.m. } \{a, a \wedge b\} \quad \text{by (2)}$$
$$= a \vee (a \wedge b) \quad \text{by (1)}$$

$= a$ by (4)

$\Rightarrow$ l.c.m. $\{a, g.c.d. \{a_1 b\}\} = 1$

and g.c.d. $\{a, l.c.m. \{a_1 b\}\}$

$= g.c.d. \{a, aVb\}$ by (1)

$= a\Lambda (aVb)$ by (2)

$= a$ by (3)

$\Rightarrow$ g.c.d. $\{a, l.c.m. \{a, b\}\} = a$ ■

Q. 24. Let (L, ≤) be a lattice and Λ and V denote the operations of meet and join in L. Then for any a, b ε L, prove that

(1) $a \le b \Leftrightarrow a\Lambda b = a$

(2) $a \le b \Leftrightarrow aVb = b$

Sol. Let (L, ≤) be lattice and operations V and Λ join and meet defined by

$aVb = \sup \{a, b\}$ ...(1)

$a\Lambda b = \inf \{a, b\}$ ...(2)

(1) Suppose that $a \le b$ then we have to show that

$a\Lambda b = a$

Since ≤ is reflexive

$\Rightarrow$ $a \le a$

Given that $a \le b$

$\Rightarrow$ a is lower bound of a and b

$\Rightarrow$ $a \le \inf \{a, b\}$

$a \le a\Lambda b$ ...(3) by (2)

Since $a\Lambda b$ is infimum of {a, b}, we have

$a\Lambda b \le a$ ...(4)

By (3) & (4) $a = a\Lambda b$ by anti–symmetric

$\therefore$ for $a \le b \Rightarrow a\Lambda b = a$ ....(A)

Now, suppose that $a\Lambda b = a$. then we have to show that $a \le b$.

Since $a\Lambda b = \inf \{a, b\}$

$\Rightarrow$ $a\Lambda b \le b$

$\Rightarrow$ $a \le b$ ($\because a\Lambda b = a$

$\therefore$ for $a\Lambda b = a \Rightarrow a \le b$ ...(B)

By (A) & (B)

$a \le b \Leftrightarrow a\Lambda b = a.$

(2) Suppose that $a \le b$ then we have to show that

$aVb = b$

Since ≤ is reflexive

$\Rightarrow$ $b \le b$

Since $a \le b$ (Given that)

$\Rightarrow$ b is an upper bound of a and b

$\sup \{a_1 b\} \le b$

$aVb \le$ ...(5) by (1)

Again by (1)

$aVb = \sup \{a, b\}$

$\Rightarrow$ $b \le aVb$ ...(6)

By (5) & (6) $aVb = b$ by anti–symmetric

$\therefore$ for $a \le b \Rightarrow aVb = b$ ...(A)

Now suppose that $aVb = b$ then we have to show that

$a \le b$

By (1) $aVb = \sup \{a, b\}$

$\Rightarrow$ $a \le aVb$

$a \le b$ ($\because aVb = b$ given)

$\therefore$ for $aVb = b \Rightarrow a \le b$ ...(B)

By (A) & (B)

$a \le b \Leftrightarrow aVb = b$ ■

Objectives

1. If A {1, 3, 5} and R {(1, 3), (1, 5), (3, 5)}, then the relation R is :

(a) Reflexive (b) Symmetric

(c) Anti-symmetric (d) Transitive

2. The number of equivalence relation on the set {1, 2, 3, 4} is :

(a) 4 (b) 15 (c) 16 (d) 24

3. The relation R = {(1, 1), (2, 3), (3, 2)} on the set A = {1, 2, 3, 4} is :

(a) Symmetric (b) Transitive

(c) Reflexive (d) Anti–symmetric

4. If A = {a, b, c} and R = {(a, a), (b, b), (b, c), (c, b)} then the relation :

(a) Reflexive and symmetric both but not transitive

(b) Symmetric and transitive both but not reflexive

(c) A partial order relation

(d) An equivalence relation.

5. If A = {1, 2, 3, 4} and R = {(1, 2), (2, 1), (4, 1), (1, 4), (3, 3)} :

(a) Reflexive (b) Symmetric

(c) Anti–symmetric (d) Transitive

6. If A {a, b, c} and R = {(a, a), (a, b), (b, c), (b, b), (c, c)} then the relation R is :

(a) Reflexive (b) Symmetric

(c) Anti–symmetric (d) Transitive

7. If A = {1, 2, 3, 4} be a set then relation R = {(1, 1), (2, 2), (3, 3), (4, 4)} is :

(a) Symmetry (b) Partial order

(c) Identity (d) Equivalence

8. **If A = {2, 4, 5, 6} and R = {(2, 4), (2, 6), (6, 2), (5, 4), (4, 5)} then relation R is :**
(a) Reflexive (b) Symmetric
(c) Transitive (d) None of these

9. **Relation ``≤'' on set of natural numbers is :**
(a) Both reflexive and symmetric but not transitive
(b) Both reflexive and transitive but not symmetric
(c) Both symmetric and transitive but not reflexive
(d) An equivalence relation.

10. **Relation ``≤'' on set of integers I is :**
(a) Reflexive, but not symmetric and transitive
(b) Transitive, but not reflexive and symmetric
(c) A partial order relation
(d) An equivalence relation.

11. **Relation `~' on set of real number R defined as $a \sim b \Leftrightarrow |a| \geq |b|$ and $a - b \geq 0$ is :**
(a) Reflexive but not symmetric and transitive
(b) Symmetric but not reflexive and transitive
(c) Transitive but not reflexive and transitive
(d) An equivalence relation

12. **Relation `⊆' on set of sets is :**
(a) Partial order relation (b) Equivalence relation
(c) Transitive and symmetric
(d) Transitive and anti symmetric

13. **If R is equivalence relation then inverse of R i.e. R^{-1} is :**
(a) Partial order relation (b) Equivalence relation
(c) Not equivalence relation (d) None of these

14. **If R and S are relations in a set A, then :**
(a) $R \cup S$ and $R \cap S$ are also relation in A but $R - S$ is not relation in A.
(b) $R - S$ and $R \cup S$ are also relation in A but $R \cap S$ is not a relation in A.
(c) $R - S$ and $R \cap S$ are also relation in A but $R \cup S$ is not a relation in A
(d) $R - S$, $R \cup S$ and $R \cap S$ are also relation in A

15. **If X = {a, b, c, d} then which of the following is not a partition of X :**
(a) P = {{a},{b,c,d}} (b) P = {{a,b},{c,d}}
(c) P = {{a},{b,c},{c,d}} (d) P = {{a},{b},{c,d}}

16. **If R and S are equivalence relation then $R \cap S$ is :**
(a) Partial order relation (b) Equivalence relation
(c) Not on equivalence relation (d) None of these

17. **If I^+ is set of all positive integers, then the relation `R' defined on I^+ by xRy ⇔ x divides y is :**
(a) reflexive and symmetric both but not transitive
(b) Symmetric and transitive both but not reflexive
(c) A partial order relation
(d) An equivalence relation

18. **In the set of integers I, the relation `≤' is :**
(a) Reflexive but neither symmetric nor transitive
(b) Transitive but neither reflexive nor symmetric
(c) A partial order relation
(d) An equivalence relation

19. **Hasse-diagram represents :**
(a) A poset (b) A Lattice
(c) A partition (d) None of these

20. **Relation ``⊆'' on power set of X is :**
(a) An equivalence relation (b) A partial order relation
(c) Not a partial order relation (d) None of these

21. **Discoverer of Hass diagram Halmut Hasse was from the countrary :**
(a) France (b) U.S.A.
(c) Germany (d) Britain

22. **Number of elements in chain is called :**
(a) Height (b) Length (c) Width (d) Vertex

23. **Number of operation in lattice is :**
(a) 1 (b) 2 (c) 3 (d) 0

24. **In relation R two elements a and b are said to be comparable. One statement is false :**
(a) $(a_1b) \varepsilon R$ (b) $(a_1b) \varepsilon R$ (c) $(b_1a) \varepsilon R$ (d) a = b

25. **Greatest element in poset (P{a, b, c}, ⊆) is**
(a) 0 (b) 1 (c) ∞ (d) {a, b, c}

26. **Least element in poset (P(a, b, c), ⊆) is :**
(a) 0 (b) 1 (c) ∞ (d) ϕ

27. **``The dual of any theorem of lattice is theorem'' this principle is known as :**
(a) Idempotent law (b) Distributive law
(c) Absorption law (d) Principle of duality

28. **Let 1 be upper bound of lattice L, then for each a ε L,**
(a) aVl = o (b) aVl = a
(c) aVl = l (d) None of these

29. Let 1 be an upper bound of lattice L then for eac a ε L:
(a) $a \wedge 1 = o$ (b) $a \wedge 1 = a$
(c) $a \wedge 1 = 1$ (d) None of these

30. Let o be a lower bound of lattice L, then for each a ε L :
(a) $a \wedge o = o$ (b) $a \wedge o = a$
(c) $a \wedge o = 1$ (d) None of these

31. Let o be the lower bound of lattice L, then for each a ε L :
(a) $a \vee o = o$ (b) $a \vee o = 1$
(c) $a \vee o = a$ (d) None of these

32. Let L be a lattice then for each a, b ε L :
(a) $a \wedge (a \vee b) = b$ (b) $a \wedge (a \vee b) = 0$
(c) $a \wedge (a \vee b) = 1$ (d) $a \wedge (a \vee b) = a$

33. Let L be a lattice then for each a, b ε L.
(a) $a \vee (a \wedge b) = a$ (b) $a \vee (a \wedge b) = b$
(c) $a \vee (a \wedge b) = 0$ (d) $a \vee (a \wedge b) = 1$

34. If R is an partial ordered relation and R^{-1} be inverse fo R then R^{-1} is :
(a) Equivalence relation (b) Partial order relation
(c) Not partial order relation (d) None of these

35. Let I^+ is the set of all positive integers, then the relation R defined on I^+ by xRy ⇔ x divides y is
(a) A partial order relation (b) An equivalence relation
(c) Total relation (d) None of these

36. Let (L, ≤) be a lattice, for a, b ε L and a ≤ b then
(a) $a \wedge b = a$ (b) $a \wedge b = b$
(c) $a \wedge b = 1$ (d) $a \wedge b = o$

37. Let (L, ≤) be a lattice, for a, b ε L, and a ≤ b then
(a) $a \vee b = a$ (b) $avb = b$
(c) $a \vee b = 1$ (d) $a \vee b = o$

Answer

(1) d	(2) b	(3) a	(4) b	(5) b	(6) a
(7) c	(8) d	(9) b	(10) c	(11) d	(12) a
(13) b	(14) d	(15) c	(16) b	(17) c	(18) c
(19) a	(20) b	(21) c	(22) b	(23) b	(24) b
(25) d	(26) d	(27) d	(28) c	(29) b	(30) a
(31) c	(32) d	(33) a	(34) b	(35) a	(36) a
(37) b					

□□□

Unit–II

Boolean Algebra- *Definition and properties, Switching circuits and its applications, Logic gates and circuits.*

Boolean functions- *Disjunctive and conjunctive normal forms, Bool's expansion theorem.*

Minimize the Boolen function using Karnaugh Map.

Boolean – Algebra : Let B be a non empty set and "+" and "." be two binary & "′" be unary operation then algebric structure (B, +, ., ′) is called Boolean Algebra if it satisfy the following properties :

1. **Closure property :** ∀ a, b ε B
   ⇒ (a + b) ε B and (a. b) ε B
2. **Commulative property : ∀ a, b ε B**
   (1) a + b = b + a (ii) a . b = b.a
3. **Existence of identify elements :**
   There exist two elements O and 1 in B such that
   (i) a + 0 = a ∀ a ε B (ii) a. 1 = a ∀ a ε B
4. Both identities are not equal, i.e.
   $0 \neq 1$
5. Distributive property : ∀ a, b, c ε B
   (i) (a + b). c = a.c + b.c
   (ii) a + b.c = (a + b) . (a + c)
6. Laws of complement :
   for each a ε B, ∋ a′ ε B such that
   (i) a + a′ = 1 (ii) a.a′ = 0
   then a′ is called complement of ′ a.

Basic Theorems of Boolean – Algebra :

Q.1. Prove that :[Idempotent law] ∀ a ε B
(i) a + a = a (ii) a .a = a

Proof L.H.S. of (1)
= a + a
= (a + a).1
= (a + a). (a + a′)
= a + a.a′
= a + 0 ∵ a + a′ = 1
= a by distribution
= R.H.S. ∵ a a′ = 0

L.H.S. of (2)
= a.a
= a.a + 0

= a.a + a a' ∵ aa' = 0
= a (a + a') by distribution
= a.1 ∵ a + a' = 1
= a
= R.H.S.

Q.2. Theorem ∀ a ε B

(i) a + 1 = 1 (ii) a. o = o

Sol. : L.H.S. of (i)

= a + 1
= a + (a + a') ∵ a + a' = 1
= (a + a) + a' by Associative law
= a + a' by Idempotent law
= 1 ∵ a + a' = 1
= R.H.S.

L.H.S. of (ii)

= a.0
= a.(a.a') ∵
= (a. a).a' by Associative law
= a.a' by Idempotent law
= 0 ∵ a a' = 0
= R.H.S.

Q.16. Theorem [Absorption – laws] ∀ a ε B

(i) a + (a.b) = a (ii) a.(a + b) = a

Sol. L.H.S. of (i)

= a + (a.b)
= a.1 + a.b
= a. (1 + b) by distribution
= a.1 ∵ a + 1 = 1
= a
= R.H.S.

L.H.S. of (ii)

= a. (a + b)
= (a + 0) . (a + b)
= a + 0.b by distribution
= a + 0 ∵ a.0 = 0
= a
= R.H.S.

Q.3. Theorem : (Involution – law) ∀ a ε B

(a')' = a

Sol. : L.H.S.

= (a')'
= 1.(a')'
= (a + a'). (a')'

= a. (a')' + a'.(a')' by distributic
= a. (a')' + 0 ∵ a.a' = 0
= a. (a')' + a.a' ∵ a a' = 0
= a. [(a')' + a']
= a.1 ∵ a' + a = 1
= a
= R.H.S.

Q.4. Theorem : [De–morgan's law] ∀ a, b ε B.

(i) (a + b)' = a'.b' (ii) (a.b)' = a' + b'.

Proof : (i) (a + b)' = a'.b'

that is, we have to show that complement of a + b is a'.b', that is, we have to show that

(1) (a + b) + a'.b' = 1
(2) (a + b) . a'.b' = 0

L.H.S. of (1)

= (a + b) + a'.b' by distribution
= [(a + b) + a']. [(a + b) + b'] by distribution
= [b + (a + a')]. [a + (b + b')] by Assosiative law
= [b + 1]. [a + 1] ∵ a + a' = 1
= 1.1 ∵ a + 1 = 1
= 1
= R.H.S.

⇒ (a + b) + a'b' = 1 ...(1)

Now, L.H.S. of (ii)

= (a + b). (a'b')
= a. (a'b') + b. (a'b') by distribution
= (a.a') .b + (b.b') .a' by Associative
= 0.b + 0.a' & commulative
= 0 + 0
= 0
= R.H.S.

⇒ (a + b). (a' . b') = 0 ...(2)

By (1) & (2)
(a + b)' = a'.b'
Now, We prove
(a.b)' = a' + b'

i.e. we have to show that complement of a.b is a' + b', i.e., we have to show that

(i) (a.b) + (a' + b') = 1
(ii) (a.b). (a' + b') = 0

L.H.S. of (1)

= (a.b) + (a' + b')

= (a' + b') + a.b — By Commulative
= [(a' + b') + a]. [(a' + b') + b]
= [b' + a + a']. [a' + b + b'] — by Commulative
= [b' + 1]. [a' + 1] — ∵ a + a' = 1
= 1.1 — ∵ a + 1 = 1
= 1
= R.H.S.

⇒ (a.b) + (a' + b') = 1 ...(A)

L.H.S. of (2)
= (a.b). (a' + b')
= (a.b).a' + a.b. (b')
= (a a').b + a(bb') — by Associative
= o.b + a.o — ∵ a a¹ = 0
= 0 + 0
= 0
= R.H.S.

⇒ (a.b) (a' + b') = 0 ...(B)

By (A) & (B)
$(a.b)^1 = a' + b'$

Note (1) 1' = 0
(2) 0' = 1

Q.5. Simplify the following by using Boolean Algebra :

(i) [a + (a'.b)].[a' +a.b]
(ii) a.b + [(a + b').b]'
(iii) a.b + ab' + a'b + a'b'
(iv) a.b + b.c + bc'
(v) [{(a'.b')' + c}.(a + c)]'
(vi) (a + b'). (a' + b). (a' + b')
(vii) [{a'.b')' + c} + (a' + c')]'

(i) [a + a'.b].[a' + a.b]
= [(a + a').(a + b)]. [(a' + a).(a' + b)] — By distribution
= [1.(a + b)].[1(a' + b)] — ∵ a + a' = 1
= (b + a).(b + a') — by commutative Prop.
= b + a.a' — by distribution
= b + 0'
= b — Ans.

(ii) a.b + [(a + b').b]'
= a.b + [a.b + b'.b]' — By distribution
= a.b + (ab)' — ∵ aa' = 0
= 1 — Ans.

(iii) a.b + a.b' + a'.b + a'.b'
= a.(b + b') +a'. (b + b') — By distribution
= a + a' — ∵ a + a' = 1
= 1 — Ans.

(iv) a.b + b.c + b.c'
= a.b + b. (c + c') — by distribution
= a.b + b — ∵ a + a' = 1
= b + b.a — by commulative law
= b — by Absorption law
Ans.

(v) [{(a'.b')' + c}.(a + c)]'
= [{(a')' + (b')' + c}.(a + c)]' — by De–morgan law
= [(a + b + c). (a + c)]' — by involution law
= [(a + c) [(a + c) + b]' — by commutative law
= [(a + c)]' — by Absorption law
= a'.c' — by morgan law
Ans.

(vi) (a + b').(a' + b).(a'+b')
= (a + b') [a' + bb'] — by distribution
= (a + b').a' — ∵ a.a' = 0
= aa' + b'.a' — and commutative law
= 0 + a'.b'
= a'.b' — Ans.

(vii) [{(a'.b')' + c} + (a' + c')]'
= [{(a')'+(b')' + c + (a' + c')]' — By De–morgan law
= [a + b + c + (a' + c')]' — by involution law
= [a + a' + b + c + c']' — by commulative law
= [a + b + 1]' — ∵ a + a' = 1
= [1]' — ∵ a + 1 = 1
= 0 — ∵ 1' = 0
Ans.

Q.6. Prove that a.b + b.c + c.a = (a + b).(b + c) . (c + a)

Sol. : R.H.S.
= (a + b). (b+c). (c+a)
= (a + b).(c+b).(c+a) — by commutative law
= (a + b). [c + b.a] — by distribution

$= (a + b).c + (a + b).\ b.a$ ...,,

$= a.c + b.c + a.\ (b.a) + b,\ (b.a)$ ...,,..

$= c.a + b.c. + a,\ b + b.a$ by idempotent & commutative

$= a.b + b.c + c.a$ ...,,..

$= L.H.S.$ **Ans.**

Q.7. Prove that

$p.q.r + p.q.r' + p.q'r + p'.q.r = p.q + q.r + r\ p$

Sol. : L.H.S.

$= p.q.r + p.q.r' + p.q'.r + p'.q.r$

$= p.q.r + p.q.r + p.qr + p.q.r^1 + p.q'r + p'.q.r$

by idempotent law

$= p.q.r. + p.q.r^1 + p.q.r + p.q'r + p.q.r + p'.q.r.$

By commutative law

$= p.q.\ (r + r') + pr\ (q + q') + q.r.\ (p + p')$ By distribution

$= p.q. + p.r + q.r$ $\because\ a + a' = 1$

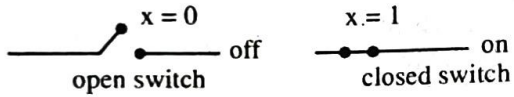
$= p.q + q.r + q.r.$ by commutative law

$= p.q. + q.r + r.p$

$= R.H.S.$ **Ans.**

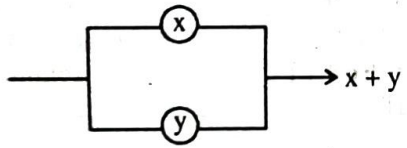
Switching Algebra

Switching is a device which is connected to an electric circuit and remains only in one of the two positions – closed or open.

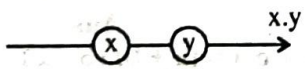


There are two types of switching circuit :

(1) In parallel



(2) In series :



Q.8. For the following switching functions, draw a simpler stwitching circuit :

(i) $P(x,y) = x.y' + x.y.$

(ii) $(f(x,y,z) = x.y.z + x.y'.z + x'.y'.z$

(iii) $f(p,q,r) = (p + q).\ (q + r).\ (r' + p')$

(iv) $f(x, y, z) = x.y'.z + (z + y).x'$

(v) $f(x, y, z) = x.z + [y.(y' + z).\ (x' + x.z')]$

(vi) $f(x,y,z) = x.y.z + (x + y)\ .\ (x + z)$

(vii) $f(x, y, z) = (x + y + z')\ .\ (x' + y)$

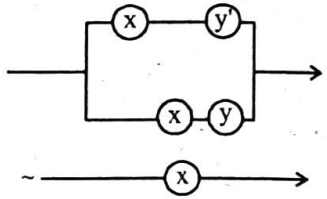
(viii) $f(x,y,z) = x.y + [z.(x' + y')]$

Sol. : (i) $f(x, y) = xy' + xy$

$= x.(y' + y)$ by distribution

$\Rightarrow\ f(x, y, z) = x$

$\therefore$ Switching circuits are as :



(ii) $f(x,y,z) = x.y.z + x.y'.z + x'.y'.z$

$= x.yz + (x + x')\ y'z$ by distribution

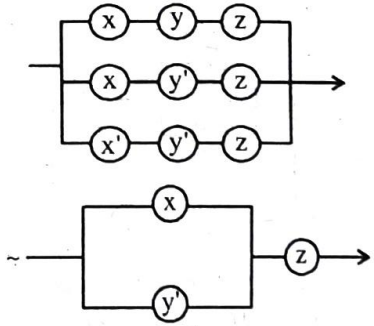
$= xyz + y'z$

$= (xy + y')z$ by distribution

$= [(x + y').(y + y')]z$ ...,,..

$\Rightarrow\ f(x, y, z) = [x + y'].z$

$\therefore$ Switching circuits are as :



Simpler

(iii) $f(p,q,r) = (p + q).\ (q + r)\ (r' + p')$

$= (q + p).(q + r)\ (r' + p')$

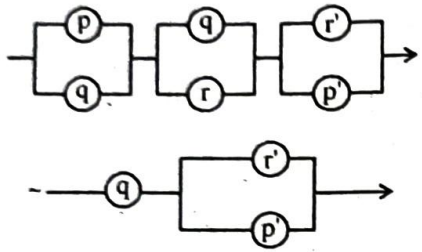
$= (q + p.r)\ (r' + p')$

$= qr' + qp' + p.r.r' + p.r.p'$

$= qr' + qp' + 0 + 0$ $\because\ a - a' = 0$

$\Rightarrow$ $f(p,q,r) = q\,(r' + p')$

$\therefore$ Switching circuits are as :



(iv) $f(x, y, z) = x.y'z + (z + y).x'$

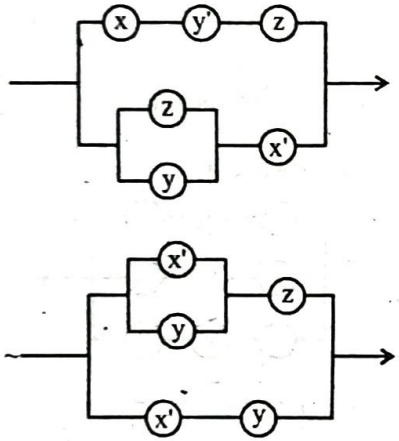
$= x.y.z + zx' + yx'$ By Distribution

$= (x' + xy')z + x'y$,,

$= [x' + x)\,(x' + y)]\,z + x'y$,,

$\Rightarrow$ $f(x, y, z) = (x' + y)\,z + x'y$

$\therefore$ switching circuits are as :



(v) $f(x, y, z) = x.z + [y.(y' + z).\,(x' + xz')]$

$= x.z + [(y.y' + y.z).(x' + x).(x' + z')]$ by distribution

$= xz + [(y.z).(x' + z')]$

$= xz + [(y.z)x' + (yz).z']$

$= xz + [x'yz + 0]$ $\because$ $aa' = 0$

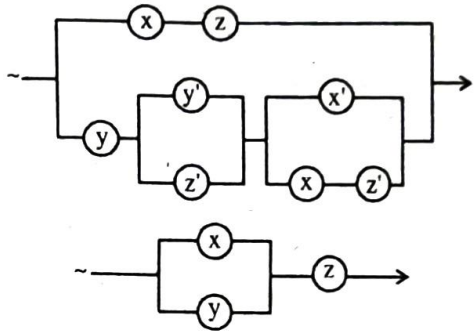
$= xz + x'yz$

$= (x + x'y)\,z$ by distribution

$= (x + x')\,(x + y)\,z$

$\Rightarrow$ $f(x, y, z) = (x + y).z.$

$\therefore$ Switching circuits are as :



Simpler.

(vi) $f(x, y, z) = x.y.z + (x + y).(x + z)$

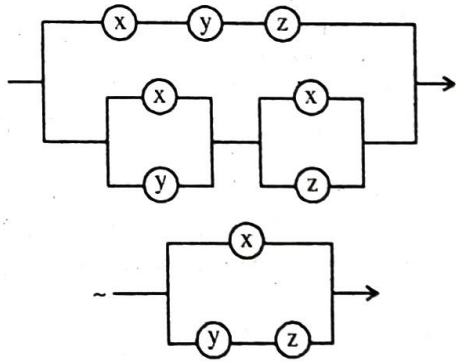
$= x.y.z + (x. + y.\,z)$ by distribution

$= x.y.z + x + y.z$

$= x + y.z + xyz$ commutative

$\Rightarrow$ $f(x, y, z) = x + yz$ by Absorption

$\therefore$ Switching circuits are as :



Simpler.

(vii) $f(x, y, z) = (x + y + z').(x' + y)$

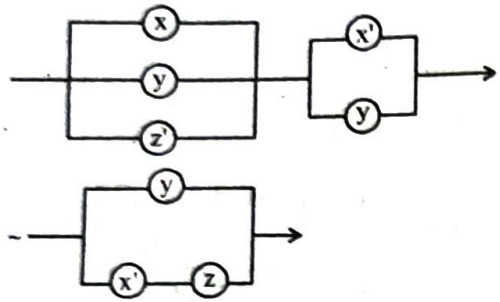
$= (y + x').(y + (x + z')$ by commutative law

$= y + x'.(x + z')$ by distribution

$= y + x'x + x'z$

$\Rightarrow$ $f(x, y, z) = y + x'z$

$\therefore$ Switching circuits are as :



(viii) $f(x, y, z) = xy + [z.(x' + y')]$

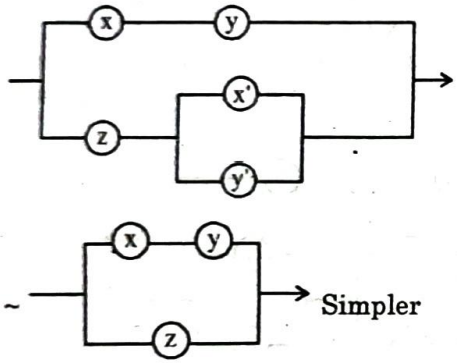
$= x.y + z\,.\,(x.y)'$ By De–morgan law

$= (x.y + z)\,.\,(x.y + (x.y)']$ By distribution

$= (x.y + z).$ \a + a' = 1

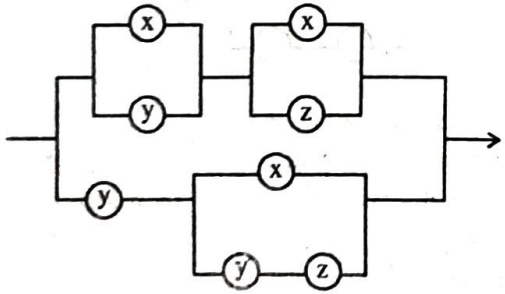
$\Rightarrow \quad f(x,y,z) = x.y + z$

∴ Switching circuits are as :



Simpler

Q.9. Find the Boolean functions of each of the following circuits and design them in simplified forms :



Sol. : There are three switches, there fore

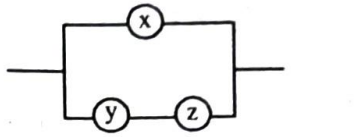
$f(x, y, z) = (x + y).\,(x + z) + y\,[x + yz]$ ...(1)

$= x + y\,z + xy + y.y.z$ By distribution

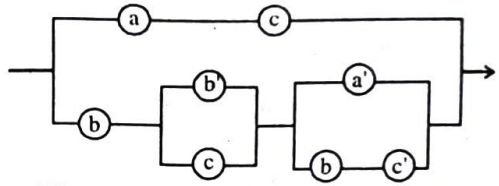
$= x + xy + yz + yz$ by commutative

$\Rightarrow \quad f(x, y, z) = x + yz$ by absoption & idempolment

So the simplified circuit is



2.



Solu : There are three switches, there fore

$f(a, b, c) = a.c + b.\,(b' + c).\,(a' + bc')$

$= a.c + (b.b' + bc).(a' + bc')$ by dist.

$= a.c + b.c\,[a' + bc']$ ∵ a a' = 0

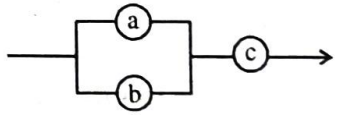
$= a.c + a'bc + bc.\,bc'$ by dist.

$= (a + a'b)\,c$ ∵ aa' = 0

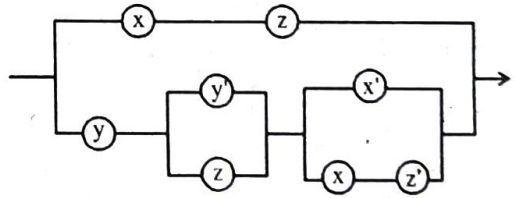
$= (a + a')\,(a + b).c$ & dist.

$\Rightarrow \quad f(a, b, c) = (a + b).c$

So, the simplified circuit is :



(3)



Sol. There are three switches, therefore

$\Rightarrow \quad f(x, y, z) = x.z + y.\,(y' + z).\,(x' + xz')$

$= xz + (y.y' + y.z).\,[(x' + x).(x' + z')]$

by distribution

$= xz + yz\,(x' + z')$

$= x.z + x'yz + yzz'$.,,,..

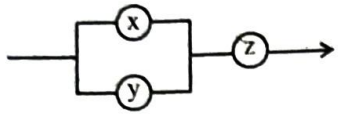
$= x.z + x'\,yz + 0$

$= (x + x'y)\,z$

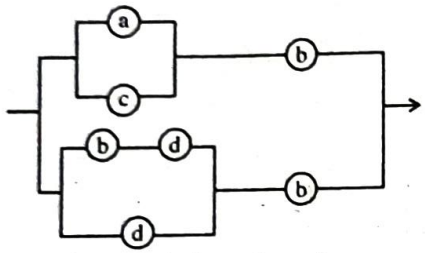
$= (x + x')\,(x + y)\,z$ By distribution

$\Rightarrow \quad f(x, y, z) = (x + y)\,z$

So the simplified circuits is



4.



There are four switches, therefore

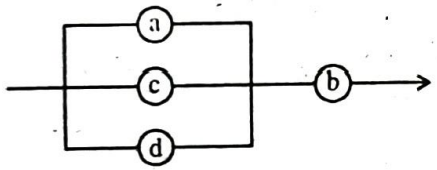
$f(a, b, c, d) = (a + c).b + (b.d + d).b$

$= (a + c).b + d.b$

$= [(a + c) + d] .b$ by distribution

$\Rightarrow f(a, b, c, d) = [a + c + d] b$

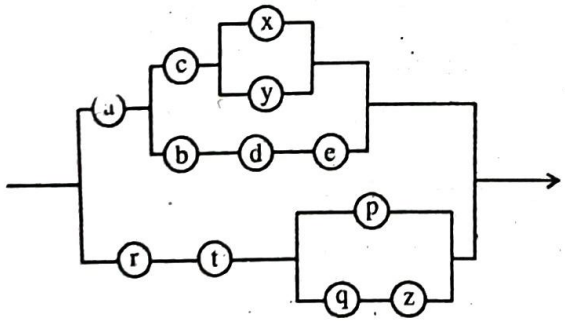
So, the simplified circuit is :



Q.10. Construct a circuit for the Boolean function :

a. [c. (x + y) + b.d.e] + r. t. (p + q.z)

Sol. : The switching circuit of the given function is :



Boolean function : Boolean function is an expression obtained by applying binary operation (+) and (.) and a unary operation of taking complement (') on the finite number of elements of a Boolean algebra (B,+,·,') where the elements of B are either the variables or constants (0 or 1).

Minimal Boolean function : A minimal Boolean function in all n–variables $x_1, x_2, \ldots\ldots x_n$ is the product of all n–letter i.e. $\bar{x}, \bar{x}_2, \bar{x}_3 \ldots\ldots \bar{x}_n$, where $\bar{x}_i = x_i$ or x'_i.

Disjunctive normal form [Canonical form]

A Boolean polymial which can be written as sum of the minimum Boolean functions, is called disjunctive normal form.

Complete disjunctive normal form [Complete canonical form]

A disjunctive normal from of n–variables is called complete disjunctive normal form if it contains 2^n – terms.

Conjunctive normal form [Dual conocial form]

A Boolean polynomial of n–variables $x_1, x_2 \ldots\ldots x_n$ is called conjunctive norm form if it is the product of distinct factors, where each factor in the sum of all distint n–letters, i.e. $v (\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \ldots\ldots + \bar{x}_n)$ when $\bar{x}_i = x_i$ or x_i'.

Complete conjunctive normal form
[Complete dual cononical form]

A conjunctive normal form in n–variables is called complete conjunctive normal form if there are 2^n distinct factors.

Q.11. Bool's expansion Theorem : State and prove Bool's expansion theorem. Or

(A) Every function without constant of a Boolean algebra can be expressed as disjunctive normal form.

(B) Every function without constant of a Boolean algebra can be expressed as conjunctive normal form.

Sol.: (A) Let $f(x_1, x_2, \ldots\ldots x_n)$ be Boolean function of n–variables without constant, it can be expressed as disjunctive normal form as follows :

(1) First use De–morgan's law as

$(a + b)' = a'.b'$ and $(a.b)' = a' + b'$

(2) Next use distributive property i.e.

$(a + b).(a + c) = a + b.c$

(3) If function f contains same terms free from x_i or x'_i then we multiply this term by $1 = x_i + x'_i$.

Repeating this process several times, every term of f is so expressed that each term contains all n–variable (x_i or x'_i, $\forall = 1, 2, 3\ldots\ldots n$).

(4) If some terms are repeated then remove it by idempotent law i.e. $a + a + a = a$.

Thus f is expressed in disjunctive normal form.

(B) Let $f(x_1, x_2, ...x_n)$ be Boolean function of n–variables without constant, it can be expressed as conjunctive normal form as follows :

(1) First we use, De–morgan's law i.e.
$(a + b)' = a'.b'$ and $(a.b)' = a' + b'$.

(2) Next we use distributive property ie
$a + b.c = (a + b).(a + c)$

(3) If function f contains some terms free from x_i or x'_i then we add this term by $0 = x_i.x_1'$
Repeating this process several times, every term of f is so expressed that each factor contains all n–variables (x_i or x_i^i).

(4) If some factors are repeated then remove it by idempotent law i.e. $a.a = a$
Thus f is expressed in conjunctive normal form.

Q.12. Express the following functions into disjunctive and conjunctive normal form:

(1) $f(x, y, z) = xy' + xz + xy$

(2) $f(x, y, z) = x.y.z + (x + y).(x + z)$

(3) $f(x, y, z) = (x + y + z).(xy + x'z)'$

Sol.: $f(x, y, z) = xy' + xz + xy$ ...(1)

(a) Disjunctive normal from by (1)

$$f(x, y, z) = xy' + xz + xy$$
$$= xy'. 1 + xz.1 + xy.1$$
$$= xy' (z + z') + xz (y + y') + xy (z + z')$$
$$= xy'z + xy'z' + xyz + xy'z + xyz + xyz'$$
$$= xyz + xy'z + xy'z' + xyz'$$
$$f(x, y, z) = xyz + xyz' + xy'z + xy'z'.$$

Which is required disjunctive normal functions.

(b) Conjunctive normal form

By (1)

$$f(x, y, z) = x.y' + x.z + x.y$$
$$= x (y + y') + xz$$
$$= x + xz$$
$$= x$$
$$= x + yy'$$
$$= (x + y) (x + y')$$
$$= (x + y + zz') (x + y' + zz')$$
$$= (x+y+z) (x+y+z') (x+y'+z) (x+y'+z')$$
$$f(x, y, z) = (x+y+z) (x+y+z') (x+y'+z) (x+y'+z')$$

Which is the required conjunctive normal form

(2) $f(x, y, z) = xyz + (x + y) (x + z)$ .....(1)

(a) disjunctive normal form

By (1) $f(x, y, z) = xyz + (x + y).(x + z)$
$= xyz + (x + yz)$
$= xyz + x + yz$
$= xyz + x(y + y') + yz (x + x')$
$= xyz + xy + xy' + xyz + x'yz$
$= xyz + xy (z + z') + xy' (x + z') + x'yz$
(By idempotent law)
$= xyz + xyz + xyz' + xy'z + xy'z' + x'yz$
$f(x, y, z) = xyz + xyz' + x'yz + xy'z + xy'z$

Which is the required disjunctive normal boolean function.

(b) Conjuctive normal form:

(i) By (1) $f(x,y,z) = xyz + (x + y).(x + z)$
$= xyz + x + yz$ by distributive law
$= x + yz + xyz$
$= x + yz$ by absorption law
$= (x + y) (x + z)$ by distributive law
$= (x + y + zz') (x + z + yy')$
$= (x + y + z) (x + y + z') (x + z + y) (x + y' + z)$
$f(x, y, z) = (x + y + z).(x + y + z').(x + y' + z)$

Which is required conjunctive normal form.

(3) $f(x,y,z) = (x + y + z).(xy + x'z)'$
$= (x + y + z) [(xy)'.(x'z)']$ By De–morgan's
$= (x + y + z) [x' + y').[(x')' + z')]$
$f(x,y,z) = (x + y + z) (x' + y') (x + z')$ ....(1)

(a) Disjunctive Normal Form

$$f(x, y, z) = (x + y + z) (x' + y') (x + z')$$
$$= (x + y + z) [(x' + y')x + (x' + y') z']$$
$$= (x + y + z) [xx' + xy' + x'z' + y'z']$$
$$= (x + y + z) [xy' + x'z' + y'z']$$
$$= (x + y + z) xy' + (x + y + z) x'z' +$$
$$(x + y + z) y'z'$$
$$= \{x.xy' + xyy' + xy'z\} + \{xx'z + x'yz' + x'zz'\}$$
$$+ \{xy'z' + yy'z' + y'zz'\}$$
$$= xy' + 0 + xy'z + 0 + x'yz' + 0 + xy'z' + 0 + 0$$
$$= xy' + xy'z + x'yz' + xy'z'$$
$$= xy' (z + z') + xy'z + x'yz' + xy'z'$$
$$= xy'z + xy'z + xy'z' + x'yz' + xy'z'$$
$$f(x, y, z) = xy'z + xy'z' + x'yz'$$

Which is the required disjunctive normal function.

(b) Conjunctive Normal Form:

By (1)

$$f (x, y, z) = (x + y + z).(x'+ y') (x + z')$$

$= (x + y + z).(x' + y' + zz').(x + z' + yy')$

$= (x + y + z)\,(x' + y' + z)\,(x' + y' + z')\,(x + y + z')\,(x + y' + z')$

$f(x, y, z) = (x + y + z)\,(x + y + z')\,(x + y' + z')\,(x' + y' + z)\,(x' + y' + z')$

Which is the required conjunctive normal form. ■

Q.13. Express the following function into disjunctive and conjunctive normal form:

(1) $f(x, y, z) = (x.y' + xz)' + x'$

(2) $f(x, y, z) = [(xy') + z')\,[z + x']'$

(3) $f(x, y, z) = (x' + y)'.(x + z)' + (y.z)'$

(4) $f(x, y, z) = (x + y)\,(x + y')\,(x' + z)$

Solu.: (1) $f(x, y, z) = (x.y' + xz)' + x'$

$= [x.(y' + z)]' + x'$

$= x' + (y' + z)' + x'$ by distributive law

$= x' + x' + (y')'.z'$ by de–morgan's law

$f(x, y, z) = x' + yz'$ .....(1) by idempotent law

(2) Disjunctive normal form

By (1) $f(x, y, z) = x' + yz'$

$= x'\,(y + y') + yz'\,(x + x')$

$= x'y + x'y' + xyz' + x'yz'$

$= x'y\,(z + z') + x'y'\,(z + z') + xyz' + x'yz'$

$= x'yz + x'yz' + x'y'z + x'y'z' + xyz' + x'yz'$

$f(x, y, z) = xyz' + x'yz + x'yz' + x'y'z + x'y'z'$

Which is the required disjunctive normal form.

(b) Conjunctive normal form :

by (a) $f(x,y,z) = x' + yz'$

$= (x' + y)\,(x' + z')$

$= (x' + y + zz')\,(x' + z' + yy')$

$= (x' + y + z)\,(x' + y + z')\,(x' + y + z')\,(x' + y' + z')$

$f(x, y, z) = (x' + y + z)\,(x' + y + z')\,(x' + y' + z')$

Which is the required conjunctive normal form.

2. $f(x, y, z) = [(x'y) + z'].[z + x']'$

$= [x' + (y')' + z').[z'.(x')']$

by dist & De–morgan law

$= [x' + y + z')\,[xz']$

$= xx'z' + xyz' + xz'z'$

$f(x, y, z) = xyz' + xz'$ ....(1)

(a) Disjunctive normal form:

By (1) $f(x, y, z) = xyz' + xz'$

$= xyz' + xz'\,(y + y')$

$= xyz' + xyz' + xy'z'$

$f(x, y, z) = xyz' + xy'z'$

Which is required disjunctive normal form.

(b) Conjunctive normal form:

By (1) $f(x, y, z) = xyz'. + xz'$

$= xz' + yxz'$

$= xz'$ by absorption law

$= xz'$

$= xz' + yy'$

$= (y + xz').(y' + xz')$ by distributive law

$= (y + x)\,(y + z').(y' + x)\,(y' + z')$

$= (x + y + zz')\,(y + z' + xx')\,(x + y' + zz')\,(y' + z' + xx')$

$= (x + y + z)\,(x + y + z')\,(x + y + z')\,(x' + y + z')\,(x + y' + z)\,(x + y' + z').(x + y' + z').(x' + y' + z')$

$f(x, y, z) = (x + y + z)\,(x + y + z')\,(x + y' + z)\,(x' + y + z')\,(x' + y' + z')$

Which is the required conjunctive normal form.

(3) $f(x, y, z) = (x' + y)'.(x + z)' + (y.z)'$

$= [(x')'\,y']\,[x'.z'] + y' + z'$

$= (xy')\,(x'z') + y' + z'$

$f(x, y, z) = y' + z'$ .....(1) \ $aa' = 0$

(a) Disjunctive normal form

By (1)

$f(x, y, z) = y' + z'$

$= y'(x + x') + z'\,(x + x')$

$= xy' + x'y' + xz' + x'z'$

$= xy'\,(z + z') + x'y'\,(z + z') + xz'\,(y + y') + x'z'\,(y + y')$

$= xy'z + xy'z' + x'y'z + x'y'z' + xyz' + xy'z' + x'yz' + x'y'z'$

$= xy'z + xyz' + xy'z' + x'y'z' + x'y'z + x'yz'$

$f(x, y, z) = xyz' + xy'z + xy'z' + x'yz' + x'y'z + x'y'z'$

Which is the required disjunctive normal form

(b) Conjuctive normal form :

By (1) $f(x,y,z) = (y' + z')$

$= (y' + z') + xx'$

$f(x, y, z) = (x + y' + z')\,(x' + y' + z')$

Which is the required conjunctive normal form.

(4) $f(x, y, z) = (x + y)\,(x + y')\,(x' + z)$ ....(1)

$= (x + yy')\,(x' + z)$

$= x(x' + z)$

$= xx' + xz$

$f(x,y,z) = xz$ .....(2)

(a) **Disjunctive normal form by (2)**

$f(x,y,z) = xz$

$= xz(y + y')$

$f(x, y, z) = xyz + xy'z$

be required disjunctive normal form.

(b) Conjunctive normal form : By (1)

$f(x, y, z) = (x + y)(x + y')(x' + z)$

$= (x + y + zz')(x + y' + zz')(x' + z + yy')$

$= (x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$
$(x' + y + z)(x' + y' + z)$

$f(x, y, z) = (x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$
$(x' + y + z)(x' + y' + z)$

Which is the required conjunctive normal forms. ■

Q.14. Change the following Boolean function to disjunctive and conjunctive normal form:

(1) $f(x, y, z) = [x + (x' + y)'].[x + (y'.z')']$

(2) $f(x, y, z) = [x + (x + y')][x + (y'.z')]$

Solu. : (1) $f(x, y, z) = [x + (x' + y)'][x + (y'.z')]$

$= [x + (x')'y'][x + (y')' + (z')']$

$= [x + xy'][x + y + z]$

$f(x, y, z) = x[x + y + z]$ ....(1)

(a) Disjunctive normal form : By (1)

$f(x, y, z) = x[x + y + z]$

$= xx + xy + xz$

$= x(y + y') + xy(z + z') + xz(y + y')$

$= xy + xy' + xyz + xyz' + xyz + xy'z$

$= xy(z + z') + xy'(z + z') + xyz + xyz' + xy'z$

$= xyz + xyz' + xy'z + xy'z' + xyz + xyz' + xy'z$

$f(x,y,z) = xyz + xyz' + xy'z + xy'z'$

Which is the required disjunctive form

(b) Conjunctive normal form:

By (1)

$f(x, y,z) = x.(x + y + z) = (x + y + z)(x + yy')$

$= (x + y + z)(x + y)(x + y')$

$= (x + y + z)(x + y + zz')(x + y' + zz')$

$= (x + y + z)(x + y + z)(x + y + z')(x + y' + z)$
$(x + y' + z')$

$f(x, y, z) = (x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$

Which is the required conjunctive normal form.

2. $f(x, y, z) = [x + (x + y')].[x + (y'z')]$

$f(x, y, z) = [x + y'](x + y'z')$ .....(1)

(a) Disjunctive normal form : by (1)

$f(x, y, z) = (x + y')(x + y'z')$

$= (x + y')x + (x + y')y'z'$

$= xx + xy' + xy'z' + y'y'z'$

$= x + xy' + xy'z' + y'z'$

$= x.(y + y') + xy'(z + z') + xy'z' + y'z'(x + x')$

$= xy + xy' + xy'z + xy'z' + xy'z' + xy'z'$
$+ x'y'z'$

$= xy(z + z') + xy'(z + z') + xy'z' + xy'z'$
$+ xyz + x'y'z'$

$= xyz + xyz' + xy'z' + xy'z + xy'z' + xy'z' +$
$xyz + x'y'z'$

$f(x, y, z) = xyz + xyz' + xy'z + xy'z' + x'y'z'$

Where is the required disjunctive normal form.

(b) Conjuctive normal form: By (1)

$f(x, y, z) = (x + y')(x + y'z)$

$= (x + y')(x + y')(x + z')$

$= (x + y' + zz')(x + z' + yy')$

$= (x + y' + z)(x + y' + z')(x + y + z')(x + y' + z')$

$f(x, y, z) = (x + y' + z)(x + y + z')(x + y' + z')$

Which is the required conjunctive normal. ■

Q.15. (a) Prove that the value of complete disjunctive normal form (of three variable) is one.

(b) Prove that the value of complete conjunctive normal form (three variable) is zero.

Solu.: (a) Let $f(x, y, z)$ be the complete the disjunctive normal form of three variables i.e. :

$f(x,y,z) = xyz + xyz' + xy'z + xy'z' + x'yz + x'yz' + x'y'z$
$+ x'y'z'$

$= xy(z + z') + xy'(z + z') + x'y(z + z') + x'y'(z + z')$

$= xy + xy' + x'y + x'y'$

$= x(y + y') + x'(y + y')$

$= x + x'$

$f(x, y, z) = 1$

(b)Let $f(x, y, z)$ be the complete conjunctive normal form of three variable ie

$f(x, y, z) = (x + y + z).(x + y + z').(x + y' + z)(x + y' + z')$
$(x' + y + z)(x' + y + z')(x' + y' + z)(x' + y' + z')$

$= [(x + y) + zz'][(x + y') + zz'][(x' + y) + zz').$
$[(x' + y') + zz']$

$= (x + y)(x + y')(x' + y)(x' + y')$

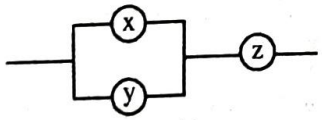
$= (x + yy')(x' + yy')$

$= x.x'$

$f(x, y, z) = 0$ ■

Objectives

1. $\forall$ a ε B.
   (a) (a')' = a' (b) (a')' = a (c) (a')' = 0 (d) (a')' = 1
2. Indempotent law is :
   (a) a + a = a (b) (a')' = a (c) a + a.b = a (d) a + 1 = 1
3. In a Boolean Algebra a + 1 is equal to :
   (a) 1 (b) q (c) 0 (d) a^1
4. In a Bollean Algebra a + ab = a is :
   (a) Idempotent law (b) Absorption law
   (c) De–morgan law (d) Associative law
5. In a Boolean Algebra, the dual of a + 1 = 1 is :
   (a) a.0 = 1 (b) a.0 = 0 (c) a.1 = 1 (d) a + 0 = 0
6. In Boolean Algebra, which of the following statement is true :
   (a) (x.y)' = x'.y' (b) x'.x' = 1
   (c) x.(x, y) = xy (d) (x + y)' = x' + y'
7. In Boolean Algebra, the value of a + a'b is :
   (a) a + b (b) a + b' (c) a' + b (d) a' + b'
8. In Boolean Algebra value of a + a + ...+a is n times :
   (a) na (b) a^n (c) a (d) n
9. In Boolean Algebra, (a + b). a' b' is :
   (a) a (b) b (c) 1 (d) 0
10. In Boolean Algebra, the dual of a.1 = a :
    (a) a.1 = a (b) a.0 = 0 (c) a + a' = 1 (d) a + 0 = 0
11. Simplified function for the switching function : f(x, y) = x + xy is :
    (a) x (b) y (c) x + y (d) x.y
12. The switching function for the following circuit is :



    (a) x.y + z (b) (x + y) . z (c) x.y.z (d) x + y + z
13. Simplified function for the switching function f(x, y, z) = xy + xy' + x'y' is :
    (a) xy (b) xy'
    (c) x + y' (d) None of these
14. The value of Boolean function in complete disjuctive normal form is :
    (a) 1 (b) 0 (c) 2^n (d) n^2

15. The number of minimal Boolean function in n–variables is:
    (a) 2^n (b) n^2 (c) 0 (d) 1
16. The value of complete conjuctive normal form is :
    (a) 1 (b) 0 (c) 2n (d) n2
17. Complement function of Boolean function xy' + x'y is :
    (a) x + y (b) x)y (c) xy + x'y' (d) None of these
18. Minimum Boolean functions in two variables x and y are:
    (a) xy' (b) xx', yy' (c) xy, x'y;', x'y, xy'(d) xy, x'y, xy'
19. The number of minimum Boolean function in three variables is :
    (a) 8 (b) 4 (c) 2 (d) 6
20. The number of fundamental Boolean functions in n–variable is:
    (a) 2^n (b) n^2 (c) $(2)^{2n}$ (d) $(2n)^2$
21. The number of fundamental Boolean functional in two variable is:
    (a) 2 (b) 4 (c) 8 (d) 16
22. Disjuctive normal form of a + a'b is
    (a) ab' + a'b' (b) ab + a'b'
    (c) ab + ab' + a'b (d) ab' + a'b + a'b'
23. Disjuctive normal form of f(x, y) = x + y' is:
    (a) x + y (b) xy + x'y
    (c) xy + xy' + x'y (d) None of these
24. Complement of Boolean function xy' + x'y + x'y' is:
    (a) x + y (b) xy (c) (x + y)' (d) (x . y)'
25. Conjunctive normal form of Boolean function f(a, b) = a + a'b is :
    (a) a + b (b) a' + b' (c) a'b' (d) ab
26. The disjunctive normal form of Boolean function f(x, y, z) = xy is :
    (a) xyz (b) x'y' (c) xyz + xyz' (d) (xy)'

Answer

1. (b)	2. (a)	3. (a)	4. (b)	5. (b)	6. (c)
7. (a)	8. (c)	9. (d)	10. (d)	11. (a)	12. (b)
13. (c)	14. (a)	15. (a)	16. (b)	17.(c)	18. (c)
19. (a)	20. (c)	21. (d)	22. (c)	23. (c)	24. (b).
25. (a)	26. (c).				

□□□

Unit–III

Graph–Definition, Type of Graphs, Subgraphs, Walk, path, circuits, connected and disconnected graph. Euler graph. Hamilton path and circuits, Dijkstra's Algorithem for shortest paths in weighted graph.

Graph :

A graph G = (V, E) consists of a non–empty set V = {v_1, v_2, v_3 ...} of objects called vertices, and another set E = {e_1, e_2, e_3, ...} whose elements are called edges, such that each edge e_k is identified with an unordered pair (v_i, v_j) of vertices. The sets V or V(G) and E or E(G) are called vertex–set and edge set respectively.

The vertices v_i, v_j associated with edge e_k are called the end vertices of e_k.

A graph with p vertices and q edges is called a (p, q) graph.

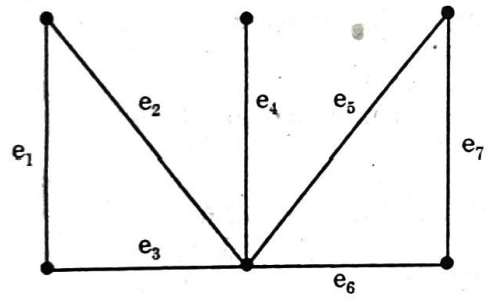


Fig. A (6, 7) graph.

Self–loop :

An edge starts and ends on the same vertex, is called self–loop.

Parallel Edges : Two or more edges starts and ends with the same pair of vertices, then they are said to be parallel edges.

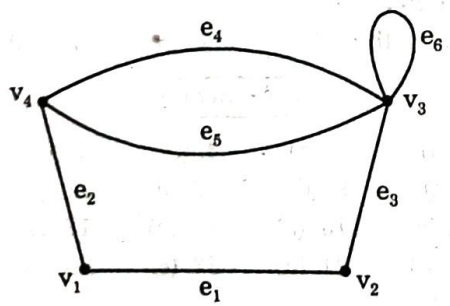


Fig. Self loop and parallel edges.

e_4 and e_5 are parallel edges.

e_6 is self loop.

Note : 1. A graph is also called a linear complex, a 1–complex or a one–dimensional complex.

2. A vertex is also reffered as a point, a node, a junction a 0–cell or a 0–complex.

3. An edge is also called a line, an arc, a branch, a 1–cell or a 1–simplex.

Incidence : When a vertex v_i is an end vertex of some edge e_j, then vi and ej are said to be incident with each other.

Adjacency : Two edges are said to be adjacent, if they are incident on a common vertex. Similarly two vertices are said to be adjacent, if they are end vertices of the same edge.

Degree of a vertex : The number of edges incident on a vertex v_i of a graph, with self loop counted twice is called the degree of v_i, and it is denoted by $d(v_i)$.

Isoleted vertex : A vertex v_i of a graph is said to be an isolated vertex, if $d(v_i)$ is zero, i.e. $d(v_i) = 0$.

Pendant vertex : A vertex v_i of a graph is said to be a pendant vertex if degree of v_i is 1, i.e. $d(v_1) = 1$.

Examples :

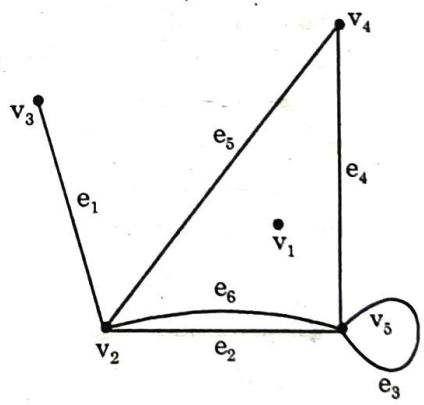


Fig. A (5, 6) graph.

(1) v_3 and e_1 are incident to each other, similarly e_2 and v_5 are incident to each other.

(2) v_1 and v_2 are adjacent to each other, similarly e_1 and e_6 are adjacent to each other, also e_1 and e_2 are adjacent to each other.

(3) e_3 is self loop.

(4) e_2 and e_6 are parallel edges.

(5) $d(v_1) = 0$, $d(v_2) = 4$, $d(v_3) = 1$
$d(v_4) = 2$, $d(v_5) = 5$.

(6) v_1 is isolated vertex, since $d(v_1) = 0$

(7) v_3 is pendent vertex, since $d(v_3) = 1$.

Order of the graph : The order of the graph is the number of the vertices in the graph. It is denoted by p or n(V).

Types of the Graph.

(1) Null–Graph : A graph G = (V, E) is called to be null graph if the set of edges E is empty i.e. $E = \phi$, while the vertex set V must not be empty i.e. $V \neq \phi$.

(2) Finite and Infinite graph : A graph G = (V, E) with a finite number of vertices as well as a finite number of edges is called a finite graph; otherwise, it is an infinite graph.

(3) Simple–Graph : A graph G = (V, E) which has neither a self loop nor parallel edges is called a simple graph.

(4) Multi–graph : A graph with parallel edges or self loop or both is called to be multi–graph.

Examples :

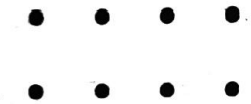


Fig. A null graph G = (8, 0) (Simple graph)

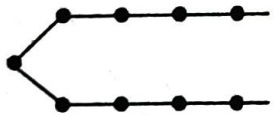


Fig. An infinite graph (Simple–graph)

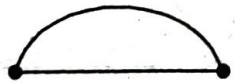


Fig. Multi–graph.



Fig. Multi–graph.

(5) Directed Graph [Diagraph] : A graph G = (V, E) is directed graph, where each edge is associated with an ordered pair of vertices called its end vertices.

If each edge is associated with the pair (u, v) of vertices, then e is said to be the (directed) edge from u to v. Thus, for directed edge e, edge e from vertex u to v, for edge e, u is the starting vertex and v is the ending vertex.

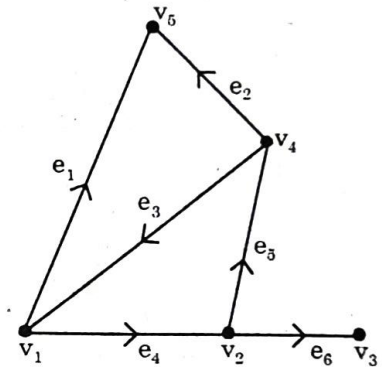


Fig. Directed Graph.

Here $e_1 = (v_1, v_5)$, $e_2 = (v_4, v_5)$, $e_3 = (v_4, v_1)$, $e_4 = (v_1, v_2)$ $e_5 = (v_2, v_4)$ and $e_6 = (v_2, v_3)$.

(6) Complete Graph : [Universal Graph] or [Clique Graph] – A graph is called to be a complete graph if there exist an edge between every pair of vertices or if there is an edge from every vertex to rest of the vertices.

The complete graph on n-vertices is denoted by K_n.

(7) Regular Graph : A graph is said to be a regular graph if degree of each vertex of the graph is same.

(8) Cycle : A graph G is said to be a cycle c_n, $n \geq 3$ consists of n–vertices and n–edges.

(9) Wheel Cycle : A graph W_n, $n \geq 3$ is said to be wheel–cycle if we induct an additional vertex to a cycle c_n and connect this vertex to each of the vertices in c_n.

(10) Biparite Graphs [Bi–graph] : A graph G = (V, E) is said to be biparite if there is a partition (V_1, V_2) of V i.e. $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$, $V_1 \neq \phi$ such that no two vertices of V_1 are adjacent and no two verties of V_2 are adjacent. Each edge of G connected a vertex of V_1 to a vertex of V_2.

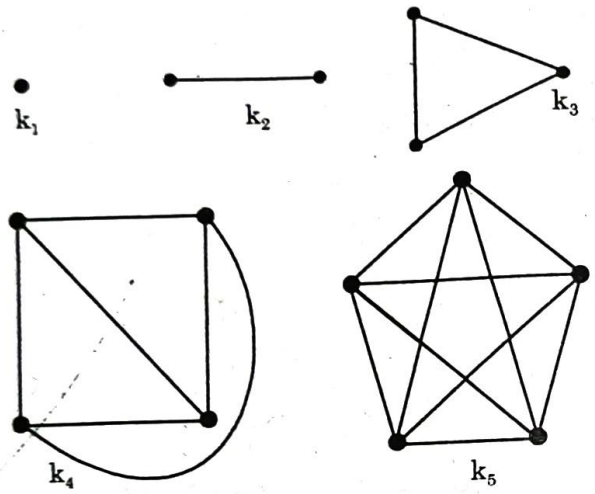


Fig. Complete Graph k_n

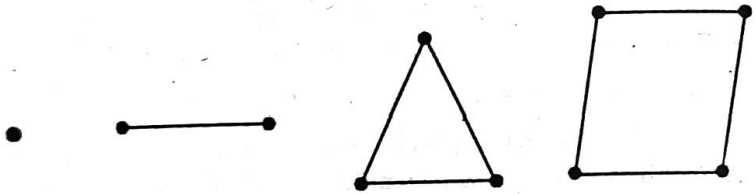


Fig. Regular-Graph

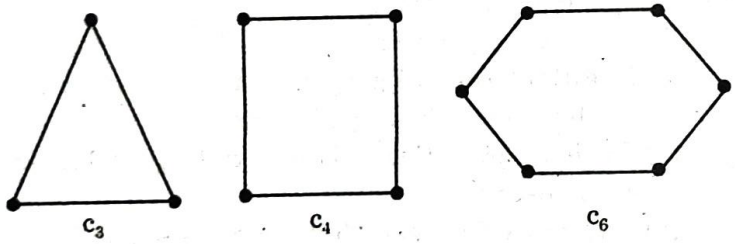


Fig. Cycle c_n

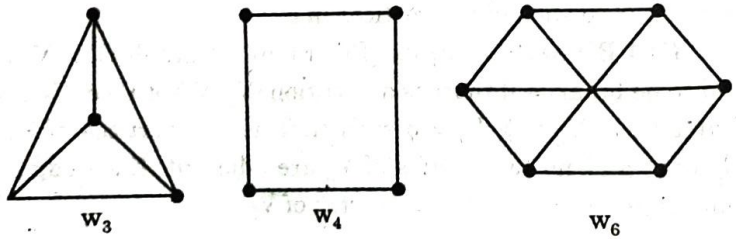


Fig. Wheel cycle W_n

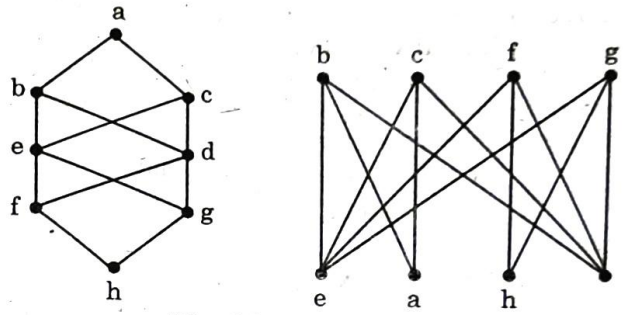


Fig. A bipartite graph.

(11) Complete Bipartite Graph : A bipartite (bigraph) G = (V, E) is said to be complete bipartite, if G contains all edges joining every vertex of V_1 with vertex of V_2, It is denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$

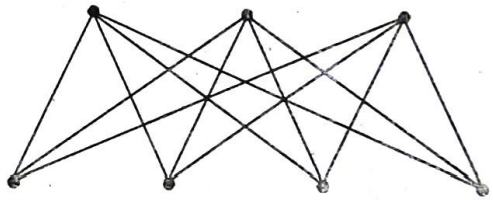


Fig : complete bigraph $K_{3,4}$.

Handshaking – Lemma

Q.1. Prove that the sum of the degree of all vertices in a graph G is equal to twice the number of edges in G.

Or

If G = (V, E) be an undirected graph with e edges and n vertices then prove that.

$$\sum_{i=1}^{n} d(v_i) = 2e.$$

Sol. Let G = (V, E) be a graph with |V| = n and |E| = e.

Since each edge in G is incident on two vertices, it contributes two to the sum of the degree of the graph. Thus the sum of degree of all the vertices in G is given by.

$$\sum_{i=1}^{n} d(v_i) = 2.e.$$

Even and Odd vertices : A vertex is said to be even vertex if the degree of the vertex is even and a vertex is said to be odd vertex the degree of the vertex is odd.

Q. 2. Prove that the vertices of odd degree (Odd vertices) in a graph is always even.

Sol. Let G = (V, E) be a graph, where E be set of edges & V be the set of all vertices of G. Let V_e and V_o denote the set of even and odd vertices in G respectively, with

$$V_e \subseteq V \text{ and } V_o \subseteq V \text{ such that}$$

$$V_e \cup V_o = V \text{ and } V_e \cap V_o = \phi.$$

Since

$$\sum_{v \in V} d(v) = \sum_{v \in V_e} d(v) + \sum_{v \in V_o} d(v) \qquad ...(1)$$

We know that

$$\sum_{v \in V} d(v) = 2|E| \equiv \text{sum of degree of all vertices of G}$$

& $$\sum_{v \in V_e} d(v) = 2k \qquad [k \in N$$

$\equiv$ Sum of even number

then by (1)

$$2|E| = 2k + \sum_{v \in V_o} d(v)$$

$$\sum_{v \in V_o} d(v) = 2\{|E| - k\} = \text{an even number}$$

Since degree of each vertices of Vo is odd & its sum is even, therefore it is only possible if the number of terms in the sum must be even

$\Rightarrow$ the number of vertices in Vo is even

$\Rightarrow$ the number of vertices of odd degree is always even.

Q. 3. Prove that the maximum number of edges in a simple graph with n-vertices is $\frac{n(n-1)}{2}$.

Sol. Let G = (V, E) be a simple graph i.e. it is without self loop and parallel edges.

Let $V = \{v_1, v_2, v_3 .... v_n\}$

The vertex v_1 can be joined to the remaining (n – 1) vertices $v_2, v_3, v_4 .... v_{n-1}, v_n$ to obtained a maximum (n – 1) edges, say (v_1, v_2), (v_1, v_3), (v_1, v_4), ... (v_1, v_n).

The vertex v_2 can be joined to the remaining (n – 2) vertices $v_2, v_4, v_5, ... v_n$ to obtained a maximum (n – 2) edges, say (v_2, v_3), (v_2, v_4) ..... (v_2, v_n)

Proceeding in this manner the vertex v_{n-1} can be joined to the remaining vertex v_n to obtained a single edge, say (v_{n-1}, v_n)

Hence the maximum number of edges in the graph G is

$= (n-1) + (n-2) + (n-3) + ... + 3 + 2 + 1$

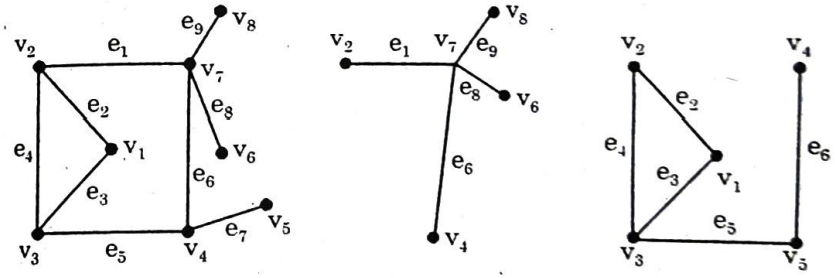
$$= \frac{n(n-1)}{2} \qquad \left[\because 1+2+...+n = \frac{n(n+1)}{2}\right.$$

Sub–Graph

A graph S is said to be sub–graph of G if all the edges and vertices of the graph S are in G, such that each edge of S has the same end vertices as in G.

Or

Let G = (V, E) be a graph then a graph $S = (V_1, E_1)$ is said to be a subgraph of the graph G if V_1 is subset of V and E_1 is subset of E such that each edge of S has the same end vertices in S as in G.



(A) Graph (B) Sub-graph (C) Sub-graph

Fig. Graph and sub-graphs.

Spanning–sub–graph :

A subgraph $S = (V_1, E_1)$ of a graph G = (V, E) is spanning subgraph of G if $V_1 = V$ i.e. both graphs have the same vertices.

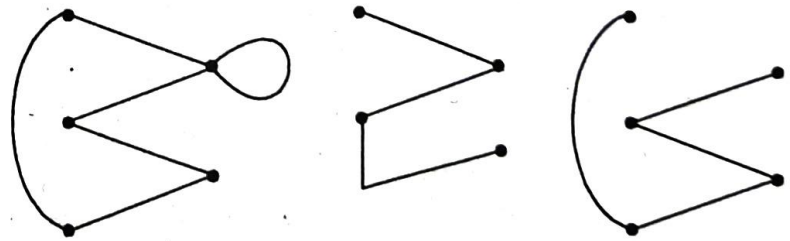


Fig. Graph and its spanning graphs.

Complement of a sub-graph

Let G = (V, E) be a graph and $G_1 = (V_1, E_1)$ be a subgraph of the graph G. Then the complement of subgraph G is subgraph say G_2 of G, i.e. $G_2 = (V_2, E_2)$ where $E_2 = E - E_1$ and V_2 containing only the vertices with which the edges in E are incident.

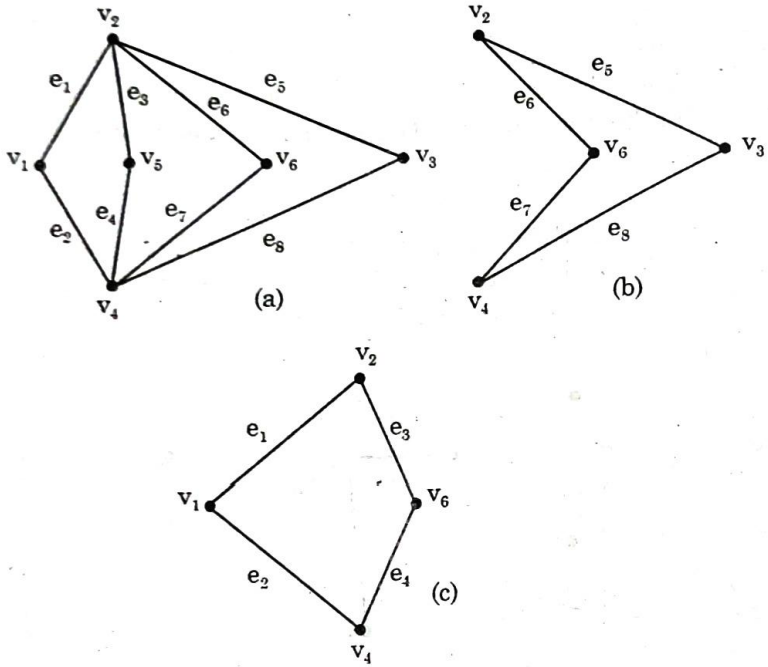


Fig. (a) Graph, (b) Subgraph & its (c) Complement

Decomposition

Let G = (V, E) be a graph then G is said to be decomposed into two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ if

$G_1 \cap G_2$ = a null graph

$G_1 \cup G_2 = G$

Say & Where and

$G_1 \cup G_2 = G_3 = (V_3, E_3)$

$G_1 \cap G_2 = G_4 = (V_4, E_4)$

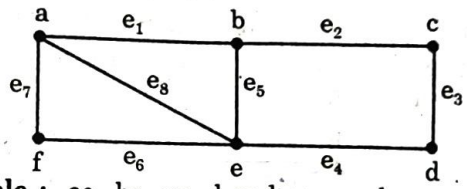
$V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$

$V_4 = V_1 \cap V_2$ and $E_4 = E_1 \cap E_2$

Walk [Chain]

A walk in a graph G = (V, E) is a finite alternative sequence of vertices and edges, beginning and ending with same or different vertices, such that no edge can appear more than once, however a vertex may.

Vertices with which walk begins and ends are called the terminal vertices of the walk.



Example : $ae_1 be_2 ce_3 d$ and $ae_8 ee_5 be_2 ce_3 de_4 ee_6 f$ are walks :

Open and closed walks : A walk with same terminal vertices (Start and end vertices) is called closed walk, otherwise it is said to be open.

Path [Simple-Path]

An an open walk, in which no vertex appears more than once, is called a path.

The number of edges in the path is called length of the path.

An edge which is not a self loop is called a path of the length one.

Cycle (circuit) [Circular path]

A closed walk in which no vertex (except its terminal vertices) appears more than once is called a cycle and it is denoted by C_k, where K is the number of vertices in the cycle.

Examples : (1) $ae_1 be_2 ce_3 d$ is open walk and a path.

(2) $ce_5 be_2 ce_3 de_4 e$ is an closed walk, i.e. a circuit or circular path.

Note : (1) In a path terminal vertices of degree one and rest of the intermediate vertices degree two only.

(2) Length of path is always one less than the number of vertices in that path.

Connected and Disconnected graph

A graph G is said to be connected, if there is at least one path betweeen every pair of vertices in G. Otherwise G is disconnected.

Weighted-Graph : A number graph is a graph in which a non-negative real number w(e) is called the weight (or length) of the edge e.

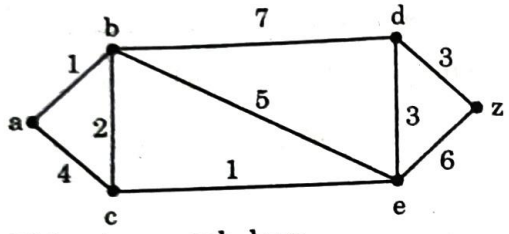
Note : The weight of edge (v_i, v_j) is the distance between v_i and v_j.

Weight of path : The weight (or length) of a path in the weighted graph G is equal to the sum of the weights of the edges in the path.

Shortest path

Shortest path between any two given vertices of a weighted graph is defined as a path of minimum weght (lenth).

Q. 4. Find the shortest path between a to z in the given weighted graph.



Sol. Let G = (V, E) be given graph, here

$V = \{a, b, c, d, e, z\}$

Step I

Let $P_1 = \{a\}$

and $R_1 = \{b, c, d, e, z\}$

Here the length of the vertices from the vertex a is

$\ell(b) = 1$

$\ell(c) = 4$

$\ell(d) = \infty$

$\ell(e) = \infty$

$\ell(z) = \infty$

Here $\ell(b) = 1 \equiv$ minimum length (weight)

$\therefore$ $P_2 = \{a, b\}$

and $R_2 = \{c, d, e, z\}$

Step II. The length of the vertices from the vertex b i.e.

$\ell(c) = \min\{$Provision length of vertex c, last minimum + length of vertex c from b$\}$

$= \min\{4, 1 + 2\}$

$\Rightarrow$ $\ell(c) = 3$

$\ell(d) = \min\{$Previous length of vertex d, last minimum + Length of vertex d from b$\}$

$= \min\{\infty, 1 + 7\}$

$\ell(d) = 8$

Similarly $\ell(e) = \min\{\infty, 1 + 5\}$

$\ell(e) = 6$

and $\ell(z) = \min\{\infty, 1 + \infty\}$

$\ell(z) = \infty$

here the minimum length $= 3 = \ell(c)$

$\therefore$ $P_3 = \{a, b, c\}$

& $R_3 = \{d, e, z\}$

Step III : The length of the vertices from vertex c, i.e.

$\ell(d) = \min\{$Previous length of d, last minimum + length of d from c$\}$

$= \min\{8, 3 + \infty\}$

$\ell(d) = 8$

Similarly $\ell(e) = \min\{6, 3 + 1\} = 4$

$\ell(z) = \min\{\infty, 1 + \infty\} = \infty$

Here the minimum length $= 4 = l(e)$

$\therefore$ $P_4 = \{a, b, c, e\}$

$R_4 = \{d, z\}$

Step IV : The length of the vertices from vertex e i.e.

$\ell(d) = \min\{8, 4 + 3\} = 7$

$\ell(z) = \min\{\infty, 4 + 6\} = 10$

here the minimum length $= 7 - l(d)$

$\therefore$ $P_5 = \{a, b, c, e, d\}$ & $R_5 = \{z\}$

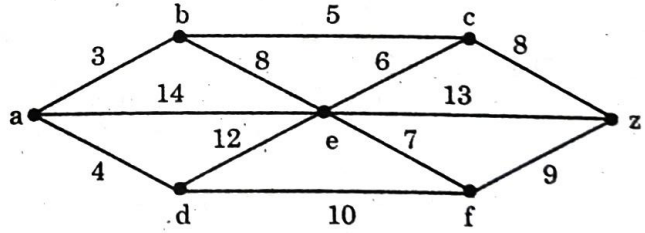
Step V : The length of the vertex from d

$\ell(z) = \min\{10, 7 + 3\}$

$\ell(z) = 10$

$\therefore$ Length of shortest path from a to z = 10, unit and a shortest path is a 1 b 2 c 1 e 6 z.

Q. 5 (a) Find the shortest path from a to z in the given weightedgraph :



Sol. Let G = (V, E) be given graph and

$V = \{a, b, c, d, e, f, z\}$

Let $P_1 = \{a\}$

and $R_1 = \{b, c, d, e, f, z\}$

Step–I : The length of vertices from the vertex a.

$\Rightarrow$ $\ell(b) = 3$

$\ell(c) = \infty$

$\ell(d) = 4$

$\ell(e) = 14$

$\ell(f) = \infty$

$\ell(z) = \infty$

here the minimum length = $3 = \ell(b)$

$\therefore$ $P_2 = \{a, b\}$

$R_2 = \{c, d, e, f, z\}$

Step–II : The length of vertices from vertex b.

$\ell(c) = \min\{$Previous length of c, last minimum + length of c from b$\}$

$= \min\{\infty, 3 + 5\}$

$\ell(c) = 8$

Similarly $\ell(d) = \min\{4, 3 + \infty\} = 4$

$\ell(e) = \min\{14, 3 + 8\} = 11$

$\ell(f) = \min\{\infty, 3 + \infty\} = \infty$

$\ell(z) = \min\{\infty, 3 + \infty\} = \infty$

here the minimum length = $4 = l(d)$

$\Rightarrow$ $P_3 = \{a, b, d\}$

$R_3 = \{c, e, f, z\}$

Step–III : The length of vertices from vertex d.

$\ell(c) = \min\{$Previous length of c, last minimum + length of c from d$\}$

$= \min\{8, 4 + \infty\} = 8$

Similarly $\ell(e) = \min\{11, 4 + 12\} = 11$

$\ell(f) = \min\{\infty, 4 + 10\} = 14$

$\ell(z) = \min\{\infty, 4 + \infty\} = \infty$

here, the minimum length = $8 = l(c)$

$\therefore$ $P_4 = \{a, b, d, c\}$

$R_4 = \{e, f, z\}$

Step–IV : The length of vertices from vertex c.

$\therefore$ $\ell(c) = \min\{$Previous length of e, last minimum + length of e from c$\}$

$= \min\{11, 8 + 6\} = 11$

Similarly $\ell(f) = \min\{14, 8 + \infty\} = 14$

$\ell(z) = \min\{\infty, 8 + 8\} = 16$

Here, the minimum length = $11 = \ell(e)$

$\therefore$ $P_5 = \{a, b, d, c, e\}$

$R_5 = \{f, z\}$

Step–V : The length of vertices from vertex e.

$\therefore$ $\ell(f) = \min\{$Previous length of f, last minimum + length of f from e$\}$

$= \min\{14, 11 + 7\} = 14$

$\ell(f) = 14$

$\ell(z) = \min\{16, 11 + 13\} = 16$

Here, the minimum length = $14 = \ell(f)$

$\therefore$ $P_6 = \{a, b, d, c, e, f\}$

$R_6 = \{z\}$

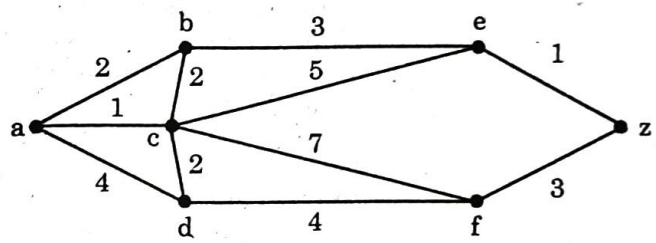
Step–VI : The length of vertex from f, i.e.

$\ell(z) = \min\{16, 14 + 9\} = 16.$

$\therefore$ The length of shortest path from a to z is 16.

& one of the shortest path is $a \to b \to c \to z$

Q. 5.(b) Find the shortest path from a to z in the given weighted graph :



Sol. Let G = (V, E) be a graph and V = {a, b, c, d, e, f, z}

Let $P_1 = \{a\}$

& $R_1 = \{b, c, d, e, f, z\}$

Step–I : The length of vertices from the vertex a, i.e.

$\ell(b) = 2$

$\ell(c) = 1$

$\ell(d) = 4$

$\ell(e) = \infty$

$\ell(f) = \infty$

$\ell(z) = \infty$

Here, the minimum length = $1 = \ell(c)$

$\therefore$ $P_2 = \{a, c\}$

$R_2 = \{b, d, e, f, z\}$

Step–II : The length of vertices from the vertex c, i.e.

$\ell(b) = \min\{$Previous length of b, last min. + length of b from c$\}$

$= \min\{2, 1 + 2\} = 2$

Similarly $\ell(d) = \min\{4, 1 + 2\} = 3$

$\ell(e) = \min\{\infty, 1 + 5\} = 6$

$\ell(f) = \min\{\infty, 1 + 7\} = 8$

$\ell(z) = \min\{\infty, 1 + \infty\} = \infty$

$\therefore$ Here, the minimum length = $2 = l(b)$

$\therefore$ $P_3 = \{a, c, b\}$

$R_3 = \{d, e, f, z\}$

Step–III : The length of vertices from the vertex b, i.e.

$\ell(d) = \min\{\text{Previous length of d, last min} + \text{length of d from b}\}$

$= \min\{3, 2 + \infty\} = 3$

and $\ell(e) = \min\{6, 2 + 3\} = 8$

$\ell(f) = \min\{8, 2 + \infty\} = \infty$

$\ell(z) = \min\{\infty, 2 + \infty\} = \infty$

Here, the minimum length = 3 = l(d)

$\therefore$ $P_4 = \{a, c, b, d\}$

$R_4 = \{e, f, z\}$

Step–IV : The length of vertices from the vertex d, i.e.

$\ell(e) = \min\{\text{Previous length of e, last min} + \text{length of e from d}\}$

$= \min\{5, 3 + \infty\} = 5$

and $\ell(f) = \min\{8, 3 + 4\} = 7$

$\ell(z) = \min\{\infty, 3 + \infty\} = \infty$

Here, the minimum length = 5 = ℓ(e)

$\therefore$ $P_5 = \{a, c, b, d, e\}$

$R_5 = \{f, z\}$

Step–V : The length of vertices from the vertex e i.e.

$\ell(f) = \min\{\text{Previous length of f, last min.} + \text{length of f from e}\}$

$= \min\{7, 5 + \infty\} = 7$

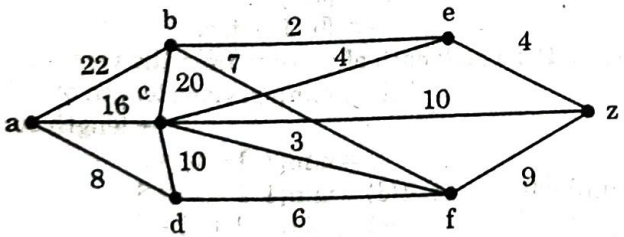
$\ell(z) = \min\{\infty, 5 + 1\} = 6$

Here, the minimum length = 6 = ℓ(z)

$\Rightarrow$ the length of shortest path from a to z is 6.

& one of the shortest path is a $\rightarrow$ c $\rightarrow$ e $\rightarrow$ z.

Q.6. (a) Find the shortest path between a and z for the following graph :



Sol. Let G = (V, E) be given graph and V = {a, b, c, d, e, f, z}

Let $P_1 = \{a\}$

& $R_1 = \{b, c, d, e, f, z\}$

Step–I : The length of vertices from the vertex a i.e.

$\ell(b) = 22$

$\ell(c) = 16$

$\ell(d) = 8$

$\ell(e) = \infty$

$\ell(f) = \infty$

$\ell(z) = \infty$

Here the minimum length = 8 = ℓ(d)

$\therefore$ $P_2 = \{a, d\}$

$R_2 = \{ab, c, e, f, z\}$

Step–II : The length of vertices from the vertex d i.e.

$\ell(b) = \min\{\text{Previous length of b, last min} + \text{length of b from d}\}$

$\ell(b) = \min\{22, 8 + \infty\} = 22$

Similarly $\ell(c) = \min\{16, 8 + 10\} = 16$

$\ell(e) = \min\{\infty, 8 + \infty\} = \infty$

$\ell(f) = \min\{\infty, 8 + 6\} = 14$

$\ell(z) = \min\{\infty, 8 + \infty\} = \infty$

Here, the minimum length = 14 = ℓ(f)

$\therefore$ $P_3 = \{a, b, f\}$

$R_3 = \{b, c, e, z\}$

Step–III : The length of vertices from vertex f i.e.

$\ell(b) = \min\{\text{Previous length of b, last min.} + \text{length of b from f}\}$

$= \min\{22, 14 + 7\}.$

$\ell(b) = 21$

& $\ell(c) = \min\{16, 14 + 3\} = 16$

$\ell(e) = \min\{\infty, 14 + \infty\} = \infty$

$\ell(z) = \min\{\infty, 14 + 9\} = 23$

Here, the minimum length = 6 = ℓ(c)

$\therefore$ $P_4 = \{a, d, f, c\}$

& $R_4 = \{b, e, z\}$

Step–IV : The length of vertices from vertex c, i.e.

$\ell(b) = \min\{\text{Previous length of b, last min.} + \text{length of b from c}\}$

$= \min\{21, 16 + 20\} = 21$

and $\ell(e) = \min\{\infty, 16 + 4\} = 20$

$\ell(z) = \min\{23, 16 + 10\} = 23$

Here, the minimum length = 20 = ℓ(e)

$\therefore$ $P_5 = \{a, d, f, c, e\}$

$R_5 = \{b, z\}$

Step–V : The length of vertices from e, i.e.

$\ell(b) = \min\{\text{Previous length of b, last min} + \text{length of b from e}\}$

$= \min\{21, 20 + 2\} = 21$

$\ell(z) = \min\{23, 20 + 4\} = 23$

Here, the minimum length = $21 = \ell(b)$

$\therefore$ $P_6 = \{a, d, f, c, e, b\}$

$R_6 = \{z\}$

Step–VI : The length of vertex z from b, i.e.

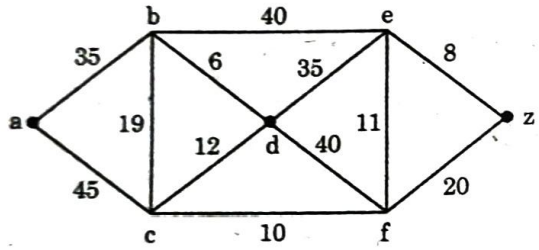
$\ell(z) = \min\{$Previous length of z, last min + length of z from b$\}$

$= \min\{23, 21 + \infty\}$

$\ell(z) = 23$

$\Rightarrow$ Length of shortest path from a to z is 23 and one of the shortest path is $a \to d \to f \to z$.

Q.6.(b) Find the shortest path between a to z for the following weighted graph.



Sol. Let G = (V, E) be given and V = {a, b, c, d, e, f, z}

Let $P_1 = \{a\}$

& $R_1 = \{b, c, d, e, f, z\}$

Step–I : The length of vertices from a i.e.

$\ell(b) = 35$

$\ell(c) = 45$

$\ell(d) = \infty$

$\ell(e) = \infty$

$\ell(f) = \infty$

$\ell(z) = \infty$

Here, the minimum length = $35 = \ell(b)$

$\therefore$ $P_2 = \{a, b\}$

$R_2 = \{c, d, e, f, z\}$

Step–II : The length of vertices from b i.e.

$\ell(c) = \min\{$Previous length of c, last min + length of c from b$\}$

$= \min\{45, 35 + 19\} = 45$

$\ell(c) = 45$

and $\ell(d) = \min\{\infty, 35 + 6\} = 41$

$\ell(e) = \min\{\infty, 35 + 40\} = 75$

$\ell(f) = \min\{\infty, 35 + \infty\} = \infty$

$\ell(z) = \min\{\infty, 35 + \infty\} = \infty$

Here, the minimum length = $41 = \ell(d)$

$\therefore$ $P_3 = \{a, c, d\}$

$R_3 = \{c, e, f, z\}$

Step–III : The length of vertices from d i.e.

$\ell(c) = \min\{$Previous length of c, last min + length of c from d$\}$

$= \min\{45, 41 + 12\} = 45$

$\ell(c) = 45$

and $\ell(e) = \min\{75, 41 + 35\} = 75$

$\ell(f) = \min\{\infty, 41 + 40\} = 81$

$\ell(z) = \min\{\infty, 41 + \infty\} = \infty$

Here, the minimum length = $45 = \ell(c)$

$\therefore$ $P_4 = \{a, b, d, c\}$

$R_4 = \{e, f, z\}$

Step–IV : The length of vertices from c, i.e.

$\ell(e) = \min\{$Previous length of e, last min. + length of e from c$\}$

$= \min\{75, 45 + \infty\} = 75$

$\ell(e) = 75$

and $\ell(f) = \min\{81, 45 + 10\} = 55$

$\ell(z) = \min\{\infty, 45 + \infty\} = \infty$

Here, the minimum length = $55 = \ell(f)$

$\therefore$ P5 = {a, b, d, c}

& $R_5 = \{e, z\}$

Step–V : The length of vertices from f i.e.

$\ell(e) = \min\{$Previous length of e, last min. + length of e from f$\}$

$= \min\{75 + 55 + 11\} = 66$

$\ell(e) = 66$

& $\ell(z) = \min\{\infty, 55 + 20\} = 75$

Here, the minimum length = $66 = \ell(e)$

$\therefore$ $P_6 = \{a, b, d, c, f, e\}$

& $R_6 = \{z\}$

Step–VI : The length of vertex z from vertex e

$\ell(z) = \min\{$Previous length of z, last min. + length of z from e$\}$

$= \min\{75, 66 + 8\}$

$\ell(z) = 75$

$\Rightarrow$ length of shortest path from a to z is 75 and one of the shortest path is $a \to c \to f \to z$.

Q. 7. Prove that a simple graph with n–vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Sol. Let G be a simple graph of n–vertices. Choose n–1 vertices v_1, v_2, v_3 .... v_{n-1} of G, we see that more than $\frac{(n-1)(n-2)}{2}$ number of edges can be drawn between these vertices. Thus, if we have more than $(n-1)(n-2)$ edges atleast one edge should be drawn between the n^{th} vertices v_n to some vertex v_i, $1 \le i \le n-1$ of G.

Hence, G must be connected.

Euler–Graph [Eulerian–Graph]

A walk covering all the edges of the graph and returning to the initial vertex is called Euler–line and graph is called Euler–graph of Eulerian–graph

Or

A graph is called Euler graph if some closed walk in a graph G contains all the edges of the graph.

Hamiltonion path and circuit

If a closed walk contains every vertex of the graph, such that the degree of every vertex is 2, then the walk is called Hamilton circuit, and if walk is open then it is said to be Hamiltonion path.

Or

A circuit that passes through each of the vertices in a graph G exactly once except the starting vertex (which is also end vertex) is called a Hamiltonion circuit.

A graph G is Hamiltonion graph if it contains a Hamiltonion circuits.

Examples

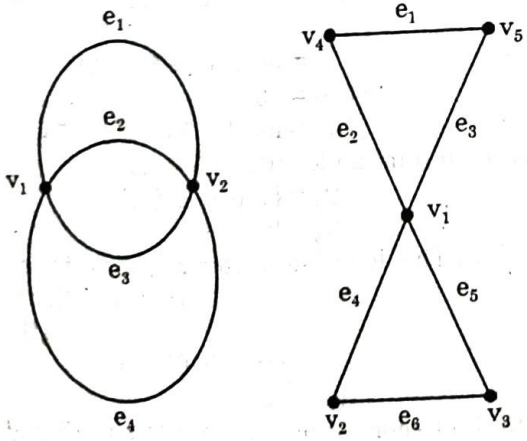


Fig. Eulerian–graph.

(1) $v_1e_1v_2e_4v_1e_2v_2e_3v_1$

and (2) $v_1e_2v_4e_1v_5e_3v_1e_4v_2e_6v_3e_5v_1$ are Eulerian–line.

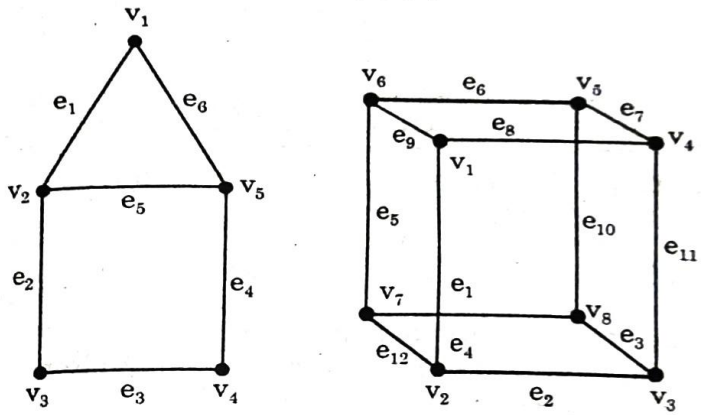


Fig. Hamiltonion–graph

(1) $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_6v_1$ and

(2) $v_1e_1v_2e_2v_3e_3v_8e_4v_7e_5v_6e_6v_5e_7v_4e_8v_1$ are Hamiltonion circuits.

Note : 1. Euler graph is always connected graph.

2. Euler graph does not contain any isolated or pendent vertex.

3. All the vertices of an Euler graph are of even degree.

4. Euler graph can be decomposed into ciruits.

5. Hamiltonion circuit is always connected graph.

Q. 7. Prove that a graph G is Euler graph if and only if all vertices of G are of even degree.

Sol. Part–I : Suppose that graph G is Euler–graph

$\Rightarrow$ It contains a Euler line (which is closed).

In tracing this walk we observe that every time the walk meet a vertex v it goes through two new edges incident on v with one we ``entered'' v and with the other `exited'. This is true for all intermediate.

Vertices of the walk with also the terminal vertex, because we `exited' and `entered' the same vertex at the beginning and end of the walk respectively.

Thus degree of each vertex of Euler–graph G is even.

Part II: Suppose that degree of all vertices of graph G is even.

Now, we construct a walk starting an an arbitrary vertex v and going through the edge of G such that no edge is traced more than once. We continue tracing as for as possible. Since every vertex

is of even degree, we can exit from each vertex we entre, the tracing cannot stop an any vertex but end with v. Since v is also of even degree we shall eventually reach v when the tracing come to an end. If this closed walk have all edges of G, then graph G is Euler graph. ■

Objectives

1. **A graph G = (V, E) is said to be finite if :**
   (a) Set V of vertices is finite.
   (b) Set E of edges is finite.
   (c) Set V of vertices and set E of edges are finite
   (d) None of these.
2. **The set E of edges of a null–graph is :**
   (a) ϕ (b) {0} (c) {ϕ} (d) None of these
3. **A simple graph contains :**
   (a) Self loop but not parallel edges.
   (b) Parallel edges but not self loop
   (c) Self loop and parallel edges
   (d) Neither self loop nor parallel edges.
4. **If v is a pendant vertex of a directed graph G, then degree of v is :**
   (a) 0 (b) 1 (c) 5 (d) 2
5. **If v is a isolated vertex of a graph G, then degree of v is :**
   (a) 0 (b) 1 (c) –1 (d) 2
6. **A vertex v is odd vertex of graph a of order n if degree of v is :**
   (a) even (b) odd (c) n (d) None of these
7. **A vertex v is even vertex of the graph G of order n, if degree of v is :**
   (a) even (b) odd (c) n (d) None of these
8. **The number of vertices of odd degree in a graph is always :**
   (a) even (b) odd (c) Prime (d) None of these.
9. **For an isolated vertex in a directed graph :**
   (a) $d^+(v) + d^-(v) = 0$ (b) $d^+(v) + d^-(v) = 2$
   (c) $d^+(v) + d^-(v) = 1$ (d) None of these
10. **For a graph G = (V, E), if |E| denotes the number edge in G, then the sum of the degrees of all vertices is equal to**
    (a) |E| (b) 2|E| (c) 3|E| (d) |E| + 1

11. **Subgraph G' is spanning graph of G if :**
    (a) At least one vertex of G is present in G'
    (b) Some vertices of G are present in G'
    (c) All vertices of G are present in G'
    (d) None of these.
12. **In a directed graph :**
    (a) Direction are fixed (b) Underlying graph is fixed
    (c) Both (a) & (b) (d) None of (a) and (b)
13. **Directed graph is finite if :**
    (a) Vertex set V is finite
    (b) Set E of edges is finite
    (c) Vertex set V and set of edges E both are finite
    (d) None of these
14. **Which of the following statement is false :**
    (a) Each subgraph is subgraph of itself
    (b) A subgraph of a subgraph of a graph G is a subgraph of G.
    (c) A single vertex in a graph G is not a subgraph of G.
    (d) A single edge in a graph G, together with its end vertices, is a subgraph of G.
15. **The maximum number of edges in a simple graph of n–vertices is :**
    (a) $\frac{n}{2}$ (b) 2n
    (c) $\frac{n(n-1)}{2}$ (d) $\frac{n(n+1)}{2}$
16. **The number of edges in a complete graph of n–vertices is :**
    (a) $\frac{n}{2}$ (b) 2n (c) $\frac{n(n-1)}{2}$ (d) $\frac{n(n+1)}{2}$
17. **If G = (V, E, f) is a graph and R^+ be the set of all positive real numbers, where $f : E \to R^+$ is :**
    (a) A simple graph (b) A multigraph
    (c) A weighted graph (d) None of these
18. **Degree of each vertex in a comple graph of n–vertices is :**
    (a) n (b) n – 1 (c) n + 1 (d) 2n
19. **Degree of each vertex in a regular graph of n–vertices :**
    (a) n (b) 2n (c) equal (d) None of these
20. **Order of a graph G is :**
    (a) Equal to the number of edges of the graph G
    (b) Equal to the number of vertices of the graph G

(c) Equal to the sum of the number of edges and number of vertices of the graph G
(d) None of these.

21. The minimum number of edges in a connected graph of n–vertices is :
(a) n (b) n – 1 (c) n + 1 (d) 2n

22. In an undirected graph, the sum of degrees of all the vertices :
(a) Must be even (b) Is twice the number of edges
(c) Must be odd (d) Both (a) and (b).

23. If V = {a, b, c, d} and E = {(a, d),)a, d), (d, a), (b, a), (c, b), (c, b), (a, c), (c, d) then G = (V, E) is :
(a) a simple graph (b) a regular graph
(c) a multi graph (d) a weighted graph

24. An undirected graph contains an Eulerian circuit if it is connected and has two vertices :
(a) Odd degree
(b) One of odd degree and other of even degee
(c) even degree
(d) None of these

25. A connected graph G is Eulerian graph if the degree of every vertex in G is :
(a) odd (b) even (c) zero (d) None of these

26. The weight of a path of a weighted graph is :
(a) Equal to number of edges in the path
(b) Equal to number of vertices in the path
(c) Equal to the sum of the weighted of the edges in the path
(d) None of these.

27. Chinese Postman Problem is related to
(a) Euler graph (b) Hamiltonion graph
(c) Tree (d) None of these

Answer

(1) c	(7) a	(13) c	(19) c	(25) b
(2) a	(8) a	(14) c	(20) b	(26) c
(3) d	(9) a	(15) c	(21) b	(27) a
(4) b	(10) b	(16) c	(22) d	
(5) a	(11) c	(17) c	(23) c	
(6) b	(12) a	(18) b	(24) a	

□□□

Unit–IV

Trees and its properties, Rooted tree, Binary tree, Spanning tree, Rank and nullity of a graph. Kruskal's Algorithem and Prim's Algorithem.

Matrix representation of graphs : *Incidence and Adjacency matrix. Cut set and its properties. Planar graph (definition) Kuratowski's two graphs.*

Tree :

A connected simple graph having no circuits is called a tree.

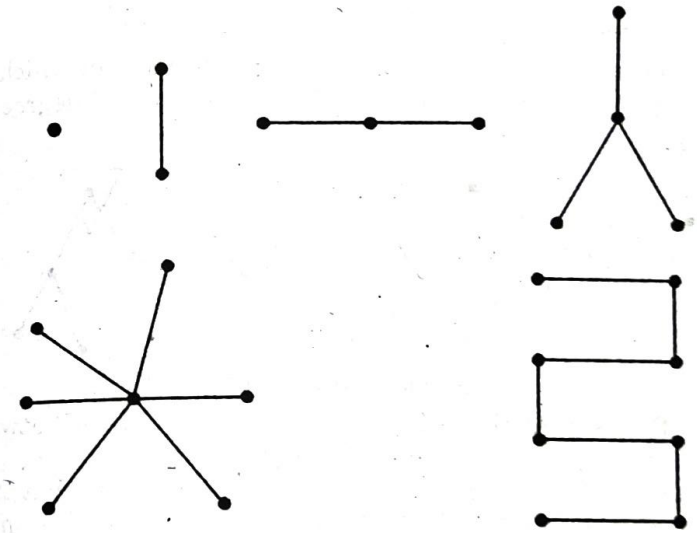


Fig. Some Trees.

Note :

1. A tree is a finite graph with at least one vertex.
2. An isolated vertex is also a tree & it is called a degenerate tree.
3. A leaf (or terminal node) in a tree is a vertex of degree one.
4. A vertex of degree more than one is called a branch node.
5. A collection of disjoint trees is said to be a forest.

Rooted Tree :

A rooted tree is a tree in which one vertex is distinguished from all other vertices. This distinguished vertex is called the root of the tree.

Any vertex of a tree may be designed the root, but we shall customarily draw a rooted tree with the root at the top (or at the bottom) and triangled or circled.

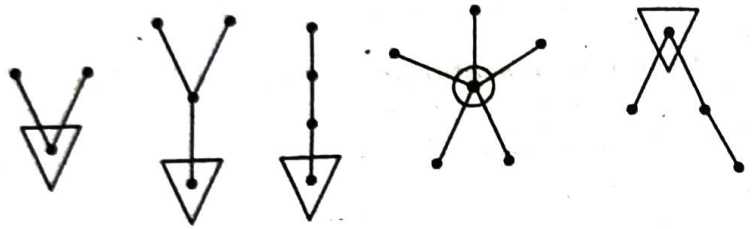


Fig. : Some rooted trees.

Binary-Tree

A binary tree is a special type of rooted tree, in which the root is of the degree two and rest of vertices either of degree one or three.

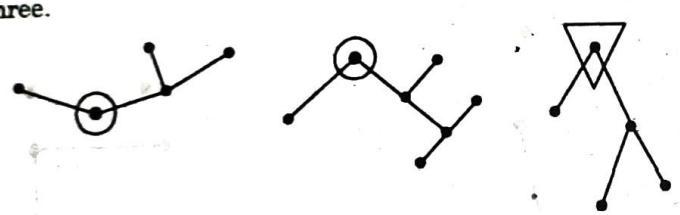


Fig. Some binary tree.

Q. 1. Prove that there is one and only one path between every pair of vertices of a tree.

Sol. Suppose that T be a tree then we have to show that there exist one and only one path between every pair of vertices in T.

Since T is a tree then T is a connected graph without circuits.

Suppose on contrary that there are two vertices u and v in T such that there are two or more than two distinct paths between u and v then the union of these paths will give us a circuit, which contradict the fact that T is a tree. Hence there is one and only one path between every pair of vertices in T. ■

Q. 2. If in a graph G there is one and only one path between every pair of vertices then G is a tree.

Sol. Suppose that G is a graph in which there is one and only one path between every pair of vertices, then we have to show that G is a tree.

Since there exists a path between every pair of vertices of G therefore G is connected.

Suppose on contrary there is a circuit in G, then there is at least one pair of vertices u and v in G such that there exist two distinct paths between u and v, which contradicts the given condition. Hence G has no circuits in G therefore G is a tree. ■

Q. 3. Prove that a tree with n–vertices has (n – 1) edges.

Sol. Let T be a tree with n–vertices.

We prove this theorem by mathematical induction on the number of vertices

If n = 1, then T has no edge

$\therefore$ number of edges in T

$= 0$

$= 1 - 1$

$= n - 1$ [$\because$ n = 1

If n = 2, then T has only one edge

$\therefore$ number of edges in T

$= 1$

$= 2 - 1$

$= n - 1$ [$\because$ n = 2

Thus the theorem is true for n = 1, 2.

Now, suppose that the theorem is true for all tree having less than n–vertices ....(*)

Since, in the tree T, there is one and only path between every pair of vertices in a tree, therefore if we delete one edge e = (u, v) (say) from T, then the graph will be disconnected. Let T_1 and T_2 be its components. Since there are no circuits in T, therefore components T_1 and T_2 both are tree and have less than n–vertices.

Let $n_1 < n$ and $n_2 < n$ be the number of vertices in T_1 and T_2 respectively, with $n_1 + n_2 = n$ ....(1)

Now by (*)

Number of edges in $T_1 = n_1 - 1$

& Number of edges in $T_2 = n_2 - 1$

$\therefore$ Number of edges in T_1 and T_2 both $= n_1 - 1 + n_2 - 1$

$= n_1 + n_2 - 2$

$= n - 2$

If we replace the edge e, then the number of edges in T

$= (n - 2) + 1$

$= n - 1$

$\Rightarrow$ the theorem is holds.

Directed graph

A directed graph is said to be directed tree if it becomes a tree when the directions of the edges are ignored.

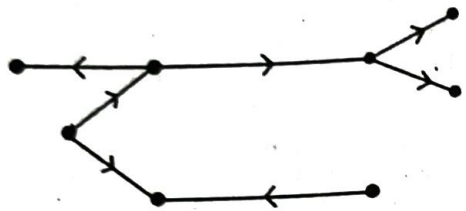


Fig. A directed tree.

Decision Tree or Sorting Tree

The sorting of the mail according to unique pin code or sorting any other thing according to any symbol or number can be considered with the help of the tree and such a tree are called decision tree.

Decision or sorting trees are labelled rooted trees which occur in applications, especially in computer programming and computer algorithms.

Distance Between Two Vertices

The distance between two vertices u and v of a connected graph G is denoted as d (u, v) and defined as the Length of shortest path between u and v i.e. the number of edges in the shortest path between u and v.

Eccentricity of a Vertex

The eccentricity of a vertex v in a graph G is denoted E(v) and is the distance from the vertex v to the farthest from v in G, i.e.

$$E(v) = \max_{v_i \in G} d(v, v_i)$$

Centre of a Graph

A centre of a graph G is a vertex with minimum eccentricity in G.

A tree may have more than one centre.

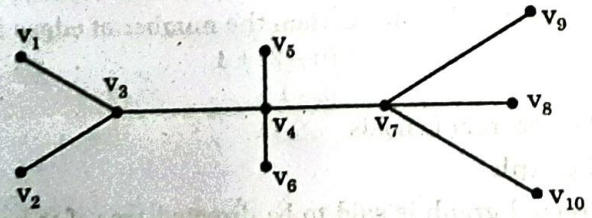


Fig. A graph (Tree) T_1.

$d(v_1, v_2) = 2$; $d(v_1, v_8) = 4$; $d(v_3, v_9) = 3$

$E(v_1) = 4$ $\quad E(v_6) = 3$

$E(v_2) = 4$ $\quad E(v_7) = 3$

$E(v_3) = 3$ $\quad E(v_8) = 4$

$E(v_4) = 2$ $\quad E(v_9) = 4$

$E(v_5) = 3$ $\quad E(v_{10}) = 10$

centre of the graph T_1 is v_4

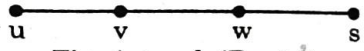


Fig. A graph (Tree) T_2

$\because$ $E(u) = 3$, $\quad E(w) = 2$

$E(v) = 2$, $\quad E(s) = 3$

$\therefore$ The graph T_2 has two centre v and w.

Note : Every tree has either one or two centres.

Diameter of the Tree

The maximum distance between two vertices of a tree is known as the diameter of tree.

Radius of a Tree

The radius of a tree is defined as the eccentricity of a centre.

diameter of $T_1 = 4$ & diameter of $T_2 = 3$

radius of $T_1 = 2$ & radius of $T_2 = 2$.

Level of a vertex in a Binary Tree

Let T be a binary tree and v be any vertex of T, then the vertex v is said to be of level ℓ if v is distance of ℓ from the root.

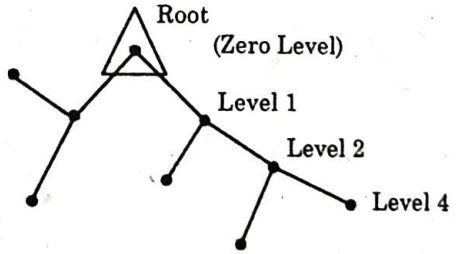


Fig. Binary Tree

Q. 4. Prove that in a tree (with two or more vertices) there are at least two pendant vertices.

Sol. Let T be a tree with n–vertices. We know that a tree with n–vertices has (n–1) edges then the sum of degree of all vertices in the tree is $2(n - 1)$.

Now 2(n – 1) degrees are to be divided amongest n–vertices so that no vertex is of degree zero (Since in a tree no vertex can have zero degree).

If all vertices be of degree 2, then the sum of degrees of all vertices will be 2n, which is not true.

If there is only one vertex of degree one, then the sum of degeree of all vertices = 2n – 1, which is also not true.

If there are two vertices of degree one, then the sum of degree of all vertices. $= 2n - 2$

$= 2(n - 1)$

Hence there are at least two pendent vertices in a tree.

The Height of a Binary Tree

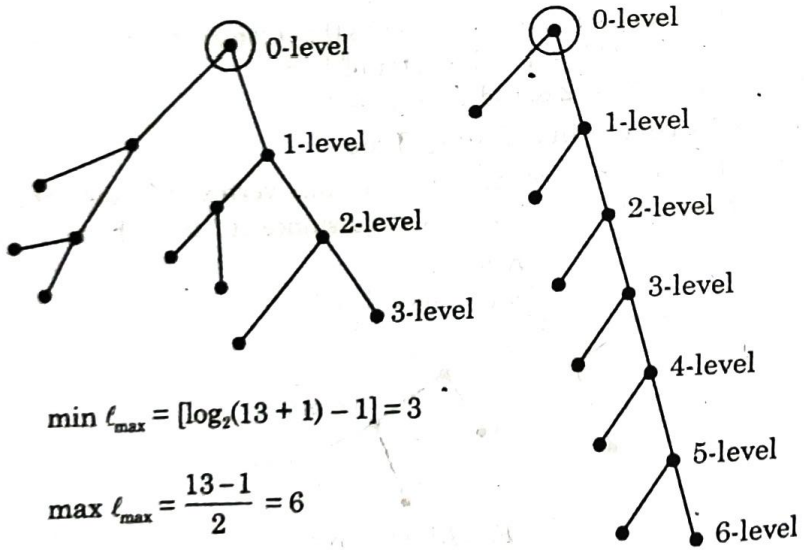
The height of a binary tree is defined as the maximum level of any vertex in the tree and is denoted by ℓ_{max}.

The minimum and maximum possible height of n–vertex binary tree are :

$$\min \ell_{max} = [\log_2(n + 1) - 1]$$

&

$$\max \ell_{max} = \frac{n-1}{2}$$



$\min \ell_{max} = [\log_2(13 + 1) - 1] = 3$

$\max \ell_{max} = \frac{13-1}{2} = 6$

Fig. Two binary tree with 13-vertices (minimum and maximum height)

Rank and Nullity fo a Graph

Let G = (V, E) be a graph, where V be the set of vertices with |V| = n and E be set of edges with |E| = e. Let K be the number of component of the graph. If graph is connected then K = 1. Then

$$\text{rank of } G = r = n - k$$

$$\text{nullity of } G = v = e - n + k$$

Note :

(1) Rank of connected graph = r = n – 1

(2) Nullity of connected graph = v = e – n + 1

(3) Rank of G = numbers of branches in any spanning tree of G

(4) Nullity of G = numbers of chords in G

(5) Rank + nullity = number of edges. ■

Q. 5. Prove that the number of vertices in a binary tree is odd.

Sol. Let T be a binary tree with n–vertices, by def of binary tree there is only one vertex of degree 2 and the remaining vertices of binary tree T are of degree one of them. Therefore the number of odd degree vertices of T is (n – 1) the number of odd degree vertices in a graph is always even.

$\Rightarrow$ (n – 1) is even number.

$\Rightarrow$ n is odd. ■

Q. 6. (a) Find the minimum height of a binary tree with 9–vertices.

(b) Find the maximal possible height of a binary tree with 13 vertices and draw the graph of the tree.

Sol. (a) We know that the minimum height of a binary tree of n–vertices is

$$\min \ell_{max} = [\log_2(n + 1) - 1] \quad ...(*)$$

Here the number of vertices in binary tree n = 9

$\therefore$ By (*)

$$\min \ell_{max} = [\log_2(9 + 1) - 1]$$

$\Rightarrow$ $\min \ell_{max} = 3$ Ans.

(b) We know that the maximum height of a binary tree of n–vertices is

$$\max \ell_{max} = \frac{n-1}{2} \quad ...(*)$$

Here the number of vertices in binary tree n = 13

$\therefore$ By (*)

$$\max \ell_{max} = \frac{13-1}{2} = 6$$

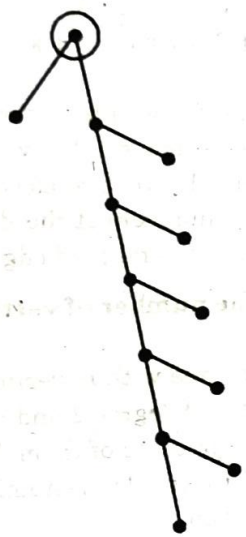
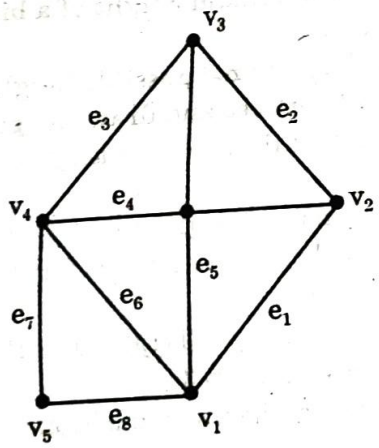


Fig. Maximum height binary tree with 13-vertices.

Q. 7. Find the rank and nullity of the following graph.



Sol. Let G be the group and n be number of vertices & e number of edges in G.

Here n = 5 and e = 8

Since graph G is connected ⇒ k = 1 = number of component

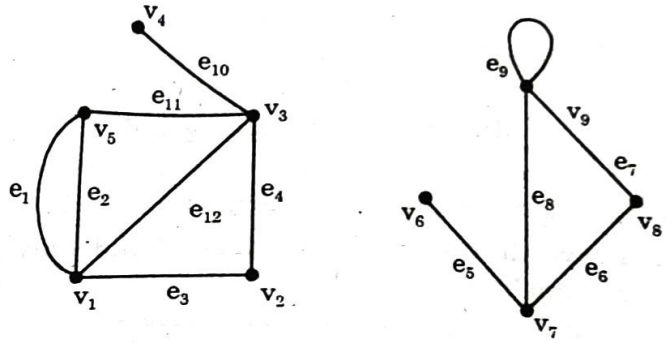
Since rank of G = r = n – k

⇒ r = 5 – 1 ⇒ r = 4

and nullity of G = v = e – n + k

v = 8 – 5 + 1 = 4

Q. 8. Find the rank and nullity of the following disconnected graph.



Sol. Let G be the graph and n be the number of vertices and e be the number of edges in G,

Here n = 9 and e = 12

Since graph G is disconnected & there is 2 component.

⇒ k = 2

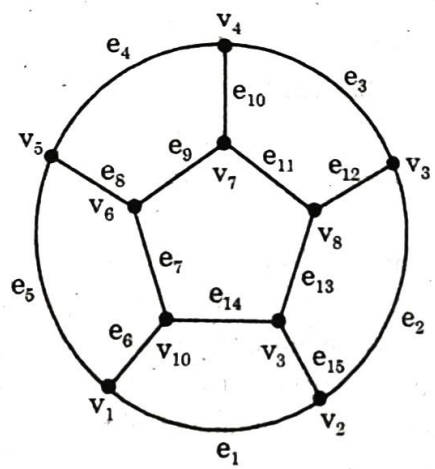
Since rank of g = r = n – k

⇒ r = 9 – 2 ⇒ r = 7

and nullity of G = v = e – n + k.

⇒ v = 12 – 9 + 2 ⇒ v = 5. ■

Q. 9. Find the rank and nullity of following graph.



Sol. Let G be the graph and n be the number of vertices and e be the number of edges in G.

Here n = 10 and e = 15

Since graph G is connected

$\Rightarrow$ $k = 1 \equiv$ number of component

Since rank of $G = r = n - k$

$\Rightarrow$ $r = 10 - 1 \Rightarrow r = 9$

and nullity of $G = v = e - n + k$

$\Rightarrow$ $v = 15 - 10 + 1 = 6$

Minimally connected graph

A connected graph G is said to be minimally connected if removal of any one edge from it disconnect the graph. ■

Q. 16. Prove that a graph is a tree if and only if it is minimally connected.

Sol. Suppose that graph G be a tree then there is one and only one path between every pair of vertices, therefore, if we remove any edge between a pair of vertices, then the remaining graph will be disconnected, and we know that a connected graph G is minimally connected if removal of any one edge from G disconnect the graph G. Hence the tree is minimally connected graph.

Conversely : Suppose that graph G is minimally connected graph. Then we have to show that G is a tree. If the graph G has circuit or parallel edges, then the removal of an edge from any circuit or from parallel edges will disconnected the graph. Therefore, the minimally connected graph has neither circuit nor parallel edges. Hence the minimally connected graph is a tree.

Spanning Tree [Skelton]

Let G be a simple graph. A spanning tree of G is a subgraph S of G that is tree containing all vertices of G.

Note :

(1) Spanning tree is a subgraph of the graph.

(2) Spanning tree is a defined only for a connected graph.

(3) There may be more than one spanning tree in a graph.

(4) A spanning tree of the graph with n–vertices always consists (n – 1) number of edges.

Branch and Chords of a Spanning Tree

Let G be a graph and S be its spanning tree then an edge in spanning tree S is called a branch of S, while an edge which is not in S, but in G is called a chord.

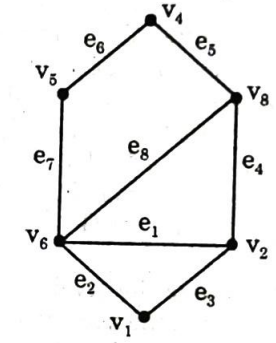


Fig. Graph G

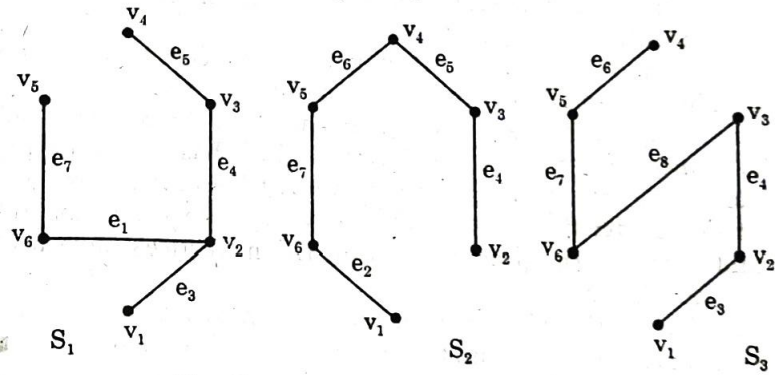


Fig. Some spanning trees of graph G.

(1) For tree S_1

(i) Branches : e_3, e_4, e_1, e_5 and e_7

(ii) Chords : E_2, e_6, e_8

(2) For Tree S_2

(i) Branches : e_2, e_4, e_5, e_6, e_7

(ii) Chords : e_1, e_3, e_8

(3) For Tree S_3

(i) Branches : e_3, e_4, e_6, e_7, e_8

(ii) Chords : e_1, e_2, e_5

Minimal Spanning Tree

Let G be a weighted graph. A minimal spanning tree of G is a spanning tree of G with minimum weight.

Methods of Construction of Minimum Spanning Tree

(1) Kruskal's Method [Kruskal–Algorithm] : In this algorithm (method), we list all edges of G in ascending order of weights.

First we select an edge of smallest weight in G which are not loops (If there is more than one edge of minimal weight, arbitrary choose one of these edges) Then for each successive step select (from all remaining edges of G) another edge of smallest weight that makes no circuit with the previously selected edges. We continue this process until n – 1 edges have been selected, and these edges from the desired minimal (shortest) spanning tree.

(2) Prim's Algorithm : Let G be a graph with n–vertices say $v_1, v_2 \ldots v_n$.

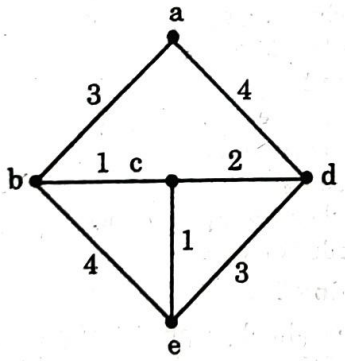
First of all, we arrange the given weights of edges of G is an n × n table. The weight of the non existent edges (i.e. between pairs of vertices where no direct edge exis) are taken as very large, say ∞. Thus we shall get a symmetric matrix of order n × n with diagonal elements as empty.

Now, we start from vertex v_1 and connect it to the vertex that has the smallest entry in first row of the table.

Suppose this vertex is v_i, join v_1 and v_i to obtain a subgraph & connect this subgraph to a vertex which has smallest entry among all entries in 1^{st} and i^{th} rows. Let his new vertex be v_j. Then consider the tree with vertices v_1, v_i and v_j as a subgraph and continue the process until all the n–vertices have been connected by n – 1 edges.

In case there is a tie for smallest entry then it can be broken arbitrary. Thus we obtain the minimum spanning tree. ■

Q. 11. Find a minimal spanning tree for the following graph using both Kruskal's and prim's methods.



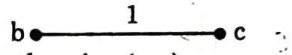
Sol. Let G be the given graph. There are five vertices in graph G therefore any spanning tree of G will have 4 edges.

1. Kruskal's Method

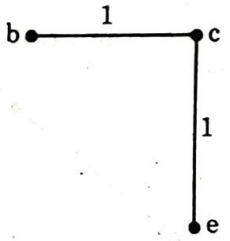
Arrange the all edges of G in ascending order of weights.

Edges :	(b, c)	(c, e)	(c, d)	(e, d)	(a, b)	(b, e)	(a, d)
Weight :	1	1	2	3	3	4	4

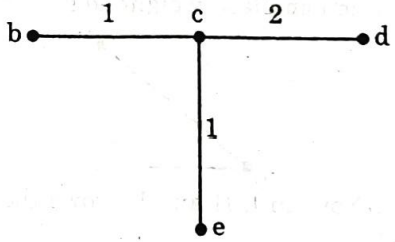
Step–I : Edges (b, c) and (c, e) are having smallest weight first choose the edge (b, c) (arbitrary)



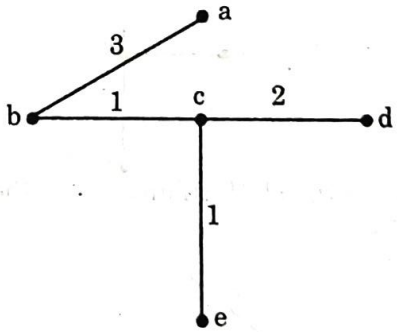
Step–II : Choose the edge (c, e)



Step–III : Next select the edge (c, d)



Step–IV : Edges (e, d) and (a, b) are having equal weight select (a, b). [If we select (e, d), then we obtain a circuit



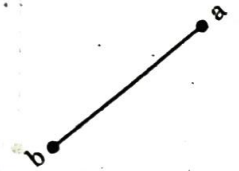
which is the required minimal spanning tree of graph G. with weight = 1 + 1 + 2 + 3 = 7 unit.

2. Prim's Method

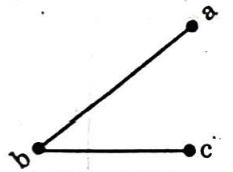
Since graph g has 5–vertices, therefore we arrange the given weight of the edges in a 5 × 5 matrix as follows :

	a	b	c	d	e
a	–	3	∞	4	∞
b	3	–	1	∞	4
c	∞	1	–	2	1
d	4	∞	2	–	3
e	∞	4	1	3	–

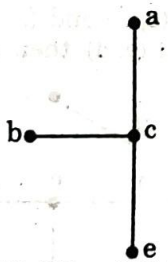
Step–I : From vertex a, in I row, the smallest (minimum) eight edge is (a, b) = 3



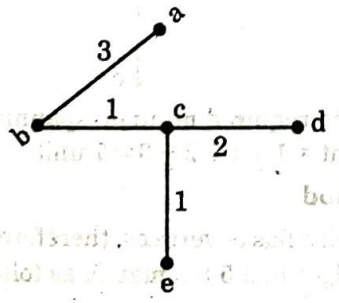
Step–II : In I and II rows
the minimum (smallest) weight edge is (b, c) = 2



Step–III : Now, in I, II and III rows the minimum weight edge is (c, e) = 1

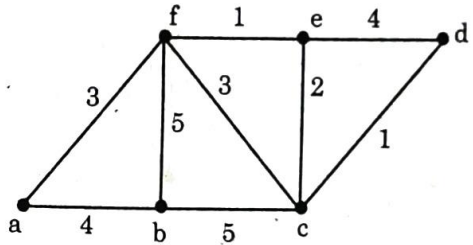


Step–IV : Now, in I, II, III and V rows the minimum weight edge is (c, d) = 2



which is the required minimum spanning tree of
3 + 1 + 1 + 2 = 7 unit weight. ■

Q. 12. Find the minimal spanning tree for then following graph using both Kruskal's and Prim's methods.



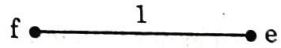
Sol. Let G be the given graph. There are six vertices in graph G therefore any spanning tree for G will have 5 edges.

1. Kruskal's Method

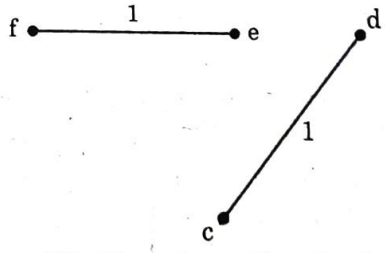
Arrange the all edges of graph G in ascending order of weights.

Edges :	(f, e)	(c, d)	(c, e)	(a, f)	(c, f)	(a, b)	(e, d)	(b, c)	(b, f)
Weights :	1	1	2	3	3	4	4	5	5

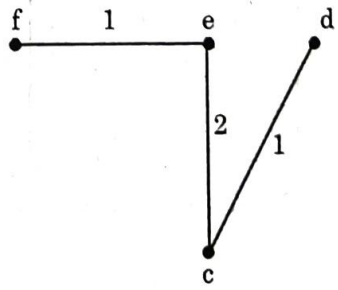
Step–I : Edges (f, e) & (c, d) are having minimum weight
First choose the edge (f, e) (arbitrary)



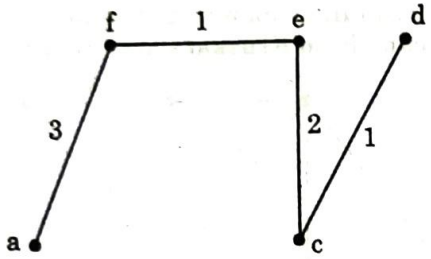
Step–II : Choose the edge (c, d)



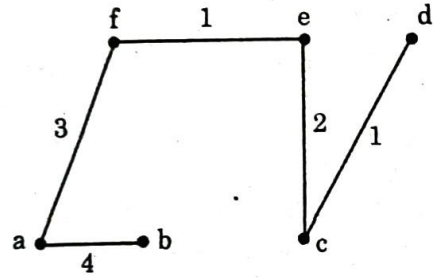
Step–III : Next choose the edge (c, e)



Step–IV : Edges (a, f) and (c, f) are having equal wieght 3 we choose (a, f) (If we select (c, f), then it makes a circuit).



Step–V : Now choose edge (a, b) = 4
(If we choose (e, d) = 4, then it makes a circuit)



Which is the required minimal spanning tree of graph with minimum weight

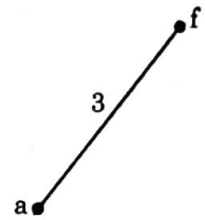
= 1 + 2 + 1 + 3 + 4 = 11 unit

2. Prim's Method :

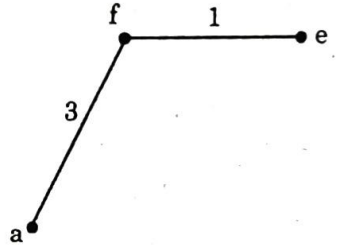
Since graph G has 6–vertices, there fore we arrange the given weight of the edges in a 6 × 6 matrix as follows :

	a	b	c	d	e	f
a	–	4	∞	∞	∞	3
b	4	–	5	∞	∞	5
c	∞	5	–	1	2	3
d	∞	∞	1	–	4	∞
e	∞	∞	2	4	–	1
f	3	5	3	∞	1	–

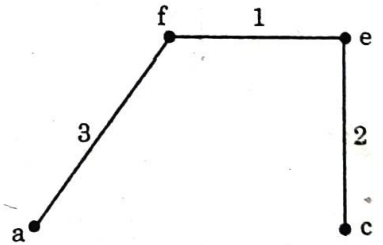
Step–I : From vertex a, In first row the minimum weight edge is (a, f) = 3.



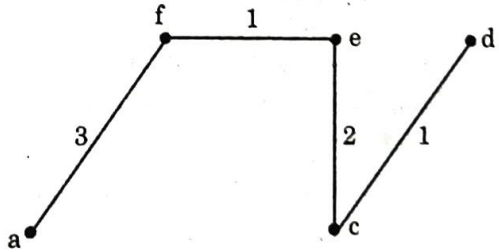
Step–II : In I and VI rows the minimum weight edge is (f, e) = 1



Step–III : In I, VI and V rows, the minimum weight edge is (e, c) = 2 i.e.

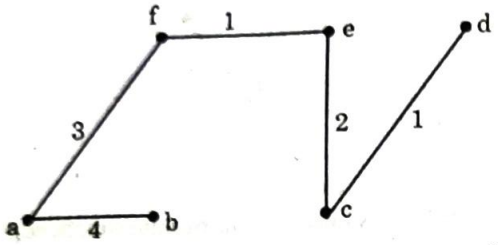


Step–IV : Now in I, VI, V and III rows, the minimum weight edge is (c, d) = 1.



Step–V : In I, VI, V, III and IV rows the minimum weight edge is (a, b) = 4

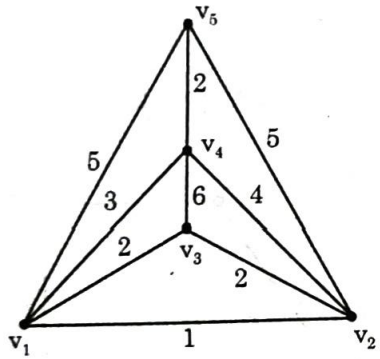
[If we taking (f, c) = 3 or (e, d) = 4 then there is obtain a circuits].



Which is the required minimal spanning tree of graph with minimum weight

$= 1 + 2 + 1 + 3 + 4 = 11$ unit. ■

Q. 13. Find a minimal spanning tree for the following graph using both Kruskal's and Prim's method :



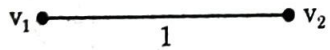
Sol. Let G be the given graph. There are five vertices in graph G therefore any spanning tree for G with have four edges.

1. Kruskal's Method :

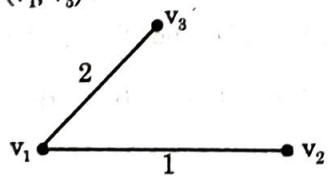
Arrange the all edges of graph G in ascending order of weights :

Edges :	(v_1, v_2)	(v_1, v_3)	(v_2, v_3)	(v_5, v_4)	(v_1, v_4)	(v_2, v_4)	(v_1, v_5)	(v_2, v_5)	(v_3, v_4)
Weights :	1	2	2	2	3	4	5	5	6

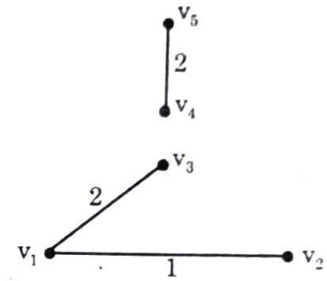
Step–I : Taking the minimum weight edge i.e. $(v_1, v_2) = 1$



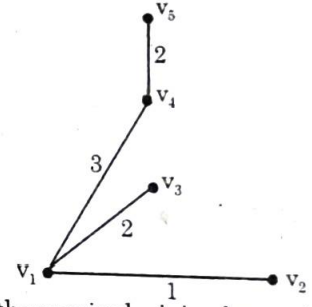
Step–II : Edges (v_1, v_2) and (v_2, v_3) are having same weight select any one say (v_1, v_3)



Step–III : Next takings (v_4, v_5).



Step–IV : Now taking edge (v_1, v_4)



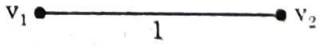
which is the one of the required minimal spanning tree with weight.

$= 1 + 2 + 3 + 2 = 8$ unit.

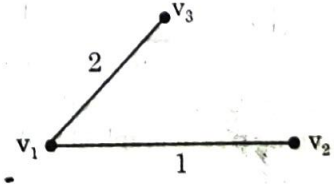
2. Prim's Method : Since graph G has 5–vertices, therefore arrange the given weight of edges in a 5×5 matrix as follows :

	v_1	v_2	v_3	v_4	v_5
v_1	–	1	2	3	4
v_2	1	–	2	4	5
v_3	2	2	–	6	∞
v_4	3	4	6	–	2
v_5	5	5	∞	2	–

Step–I : In first row, start from v_1, the minimal weight edge is $(v_1, v_2) = 1$

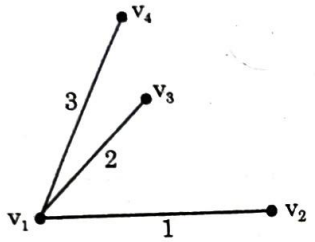


Step–II : In I and II rows, the minimal weight edge is $(v_1, v_3) = 2$.

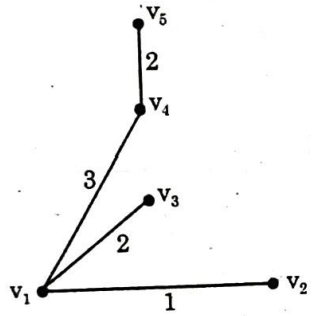


Step–III : In I, II and III rows, the minimal weight edge is $(v_1, v_4) = 3$

(If we take $(v_2, v_3) = 2$, then it makes a circuit)



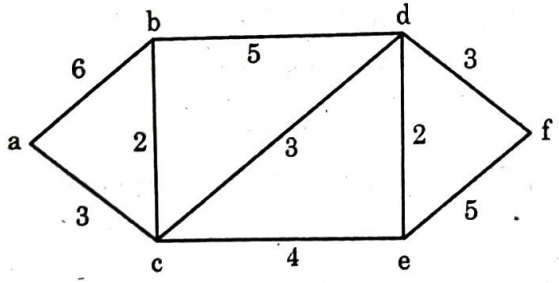
Step–IV : In I, II, III and IV rows, the minimal weight edge is $(v_2, v_5) = 2$.



Which is the one of the required minimal spanning tree with weight

$= 1 + 2 + 3 + 2 = 8$ unit.

Q. 14. Find a minimal spanning tree for the following graph using both Kruskal's and Prim's method.



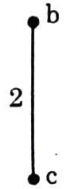
Sol. Let G be the given graph. Since given graph has 6 vertices, therefore any spanning tree of graph G will have 5 edges.

1. Kruskal's Method :

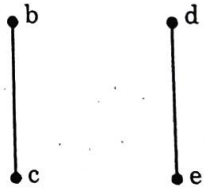
Arrange the all edges of graph G in ascending order of weights :

Edges :	(b, c)	(d, e)	(a, c)	(c, d)	(d, f)	(c, e)	(b, d)	(e, f)	(a, b)
Weights :	2	2	3	3	3	4	5	5	6

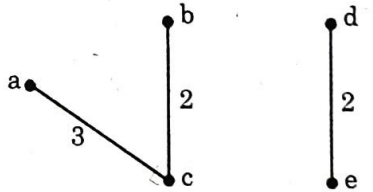
Step–I : Taking the minimum weight edge (b, c) = 2



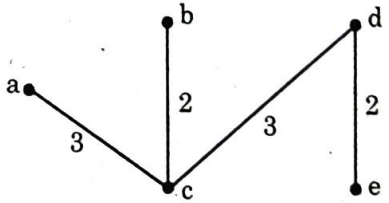
Step–II : Taking the next minimum weight edge (d, e) = 2



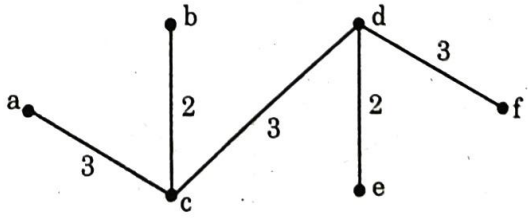
Step–III : Taking the minimum weight edge (a, c) = 3



Step–IV : Now taking the minimal weight edge (c, d) = 3



Step–V : Now taking the minimal weight edge (d, f) = 3



Which is the one of the required minimal spanning tree with minimal weight.

$= 3 + 2 + 3 + 2 + 3 = 13$ unit.

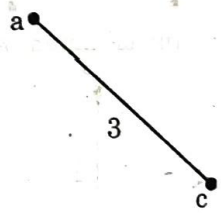
2. Prim's Method

Here the number of vertices is 6, therefore its matrix of weights is given by

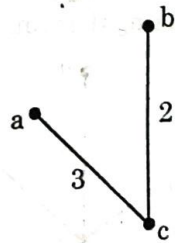
	a	b	c	d	e	f
a	–	6	3	∞	∞	∞
b	6	–	2	5	∞	∞
c	3	2	–	3	4	∞
d	∞	5	3	–	2	3
e	∞	∞	4	2	–	5
f	∞	∞	∞	3	5	–

Step–I : Start from vertex a.

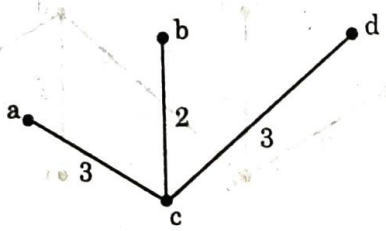
In I row, the minimum weight edge is (a, c) = 3



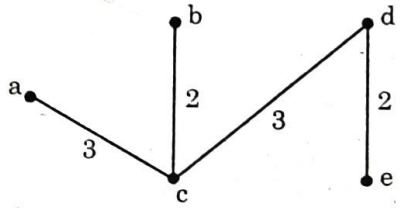
Step–II : In I and III rows, the minimal weight edge (c, b) = 2



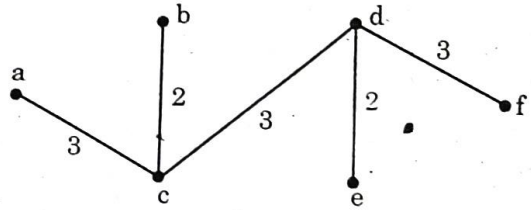
Step–III : In I, III & II rows, the minimum weight edge is (c, d) = 3



Step–IV : In I, III, II & IV rows, the minimum weight edge is (d, e) = 2.

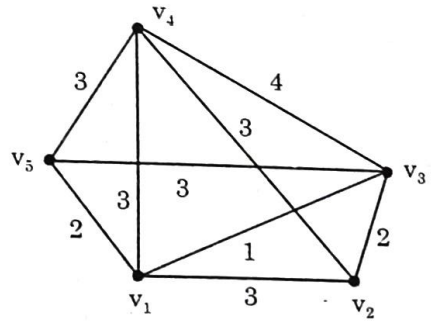


Step–V : In I, III, II, IV and V rows, the minimum weight edge is (d, e) = 3



Which is the required minimal weight spanning tree with minimum weight = 3 + 2 + 3 + 2 + 3 = 13 unit. ■

Q. 15. Find the minimal spanning tree for the following graph using both Krushkal's and Prim's methods :



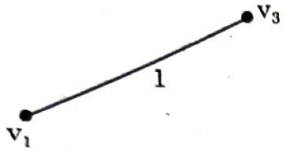
Sol. Let G be the given graph. Since given graph has five vertices, therefore any spanning tree of graph G will have four edges.

1. Kruskal's Method

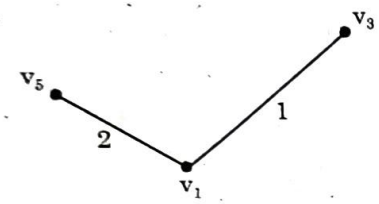
Arrange the all edges of graph G in ascending order of weights :

Edges :	(v_1, v_3)	(v_1, v_5)	(v_2, v_3)	(v_1, v_4)	(v_1, v_2)	(v_2, v_4)	(v_3, v_5)	(v_4, v_1)	(v_3, v_4)
Weights :	1	2	2	3	3	3	3	3	4

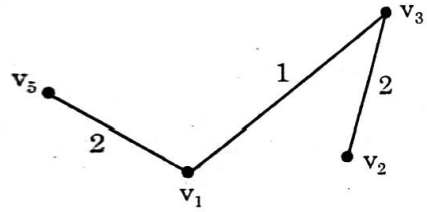
Step–I : Taking the minimum weight edge i.e. $(v_1, v_3) = 1$



Step–II : Since weight of edges (v_1, v_5) and (v_2, v_3) are have same weight, at random take any one say $(v_1, v_5) = 2$

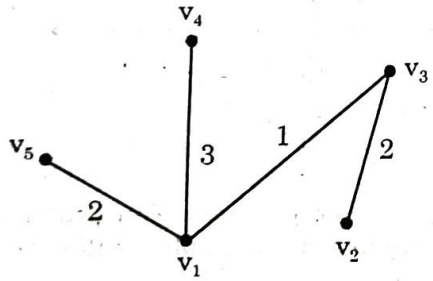


Step–III : Now taking the edge $(v_2, v_3) = 2$



Step–IV : Since weights of edges (v_1, v_4), (v_1, v_2), (v_2, v_4), (v_4, v_5) are have equal.

$\therefore$ taking (v_1, v_4)



Which is the required minimal spanning tree, with minimal weight = 2 + 3 + 1 + 2 = 8 unit.

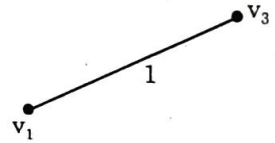
2. Prim's Method

Here the number of vertices is 5, therefore its matrix of weight is given by

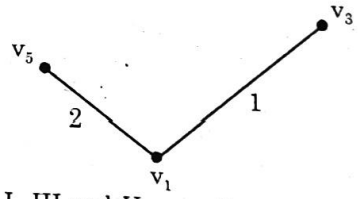
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} - & 3 & 1 & 3 & 2 \\ 3 & - & 2 & 3 & \infty \\ 1 & 2 & - & 4 & 3 \\ 3 & 3 & 3 & - & 3 \\ 2 & \infty & 3 & 3 & - \end{bmatrix} \end{array}$$

Step–I : Start from vertex v_1.

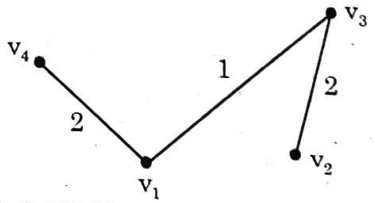
In I row, the minimum weight edge is $(v_1, v_3) = 1$



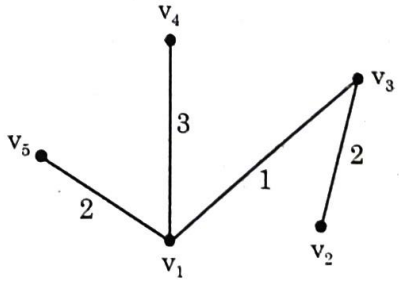
Step–II : In I and III rows, the minimum weight edge is $(v_1, v_5) = 2$



Step–III : I, III and V rows the minimum weight edge is $(v_2, v_3) = 2$.

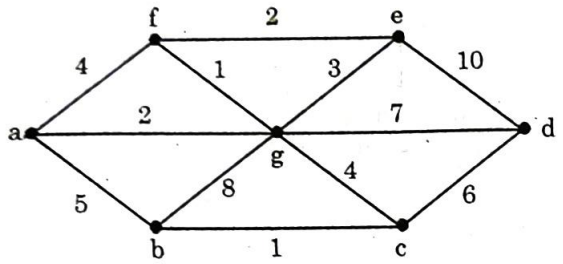


Step–IV : In I, III, V and II rows, the minimum weight edge is $(v_1, v_4) = 3$.



Which is the required minimal spanning tree with minimal weight = 1 + 2 + 2 + 3 = 8 unit. ■

Q. 16. Find a minimal spanning tree for the following graph using both Kruskal's and Prim's methods :



Sol. Let G be a graph with seven vertices, therefore any spanning tree of G has six edges.

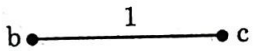
1. Kruskal's Method

Arrange the all edges of graph G in ascending order of weights.

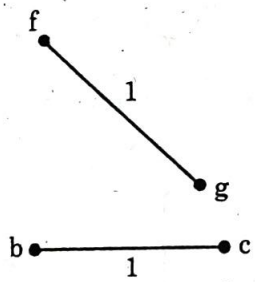
Edges :	(b, c)	(f, g)	(a, g)	(f, e)	(e, g)	(a, f)	(c, g)	(a, b)
Weight :	1	1	2	2	3	4	4	5

(c, d)	(d, g)	(b, g)	(d, e)
6	7	8	10

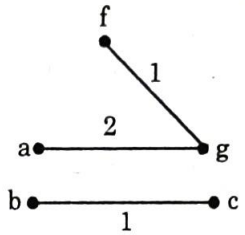
Step–I : Edges (b. c) & (f, g) are having same minimum weight, therefore arbitrary, we can choose any one edge choose the edge (b, c)



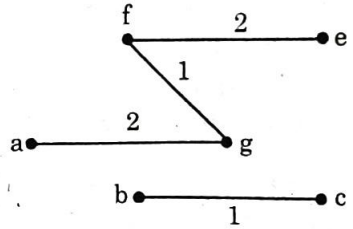
Step–II : Now choose next edge (f, g)



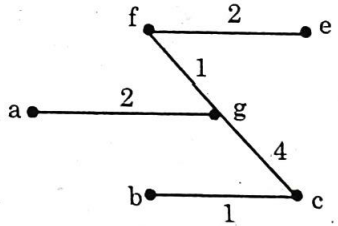
Step–III : Again, edges (a, g) and (f, e) are having same weight, taking any one say (a, g)



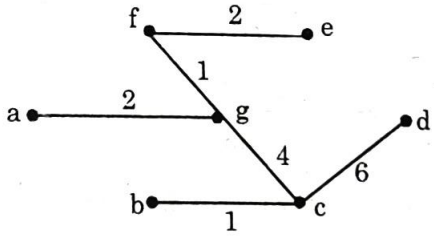
Step–IV : Now taking edge (f, e)



Step–V : Taking next minimum weight edge (c, g) = 4 (If we choose (e, g) = 3 or (a, f) = 4 then it makes a circuit]



Step–IV : Now, taking next minimum weight edge (c. d) = 6 [If take (a, b) = 5, it makes a circuit.]



Which is the required minimal spanning tree, with minimum weight = 1 + 1 + 2 + 2 + 4 + 6 = 16 unit.

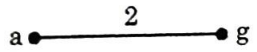
2. Prim's Method

Here the number of vertices is 7, therefore its matrix of weight of edges is given by.

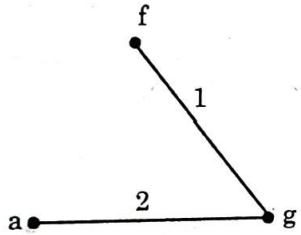
$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} - & 5 & \infty & \infty & \infty & 1 & 2 \\ 5 & - & 1 & \infty & \infty & \infty & 8 \\ \infty & 1 & - & 6 & \infty & \infty & 4 \\ \infty & \infty & 6 & - & 10 & \infty & 7 \\ \infty & \infty & \infty & 10 & - & 2 & 3 \\ 4 & \infty & \infty & \infty & 2 & - & 1 \\ 2 & 8 & 4 & 7 & 3 & 1 & - \end{bmatrix} \end{array}$$

Step–I : Start from vertex a.

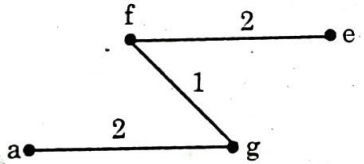
In I row the minimum weight edge is (a, g) = 2



Step–II : In I and VII rows, the minimum weight edge is (g, R) = 1

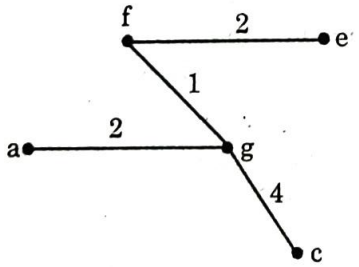


Step–III : In I, VII and VI rows, the minimum weight edge is (f, e) = 2.

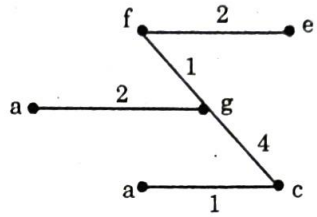


Step–IV : In I, VII, VI and V rows, the minimum weight edge is (g, c) = 4.

[If we take (e, g) = 3 or (a, f) = 4, then it makes the circuits]

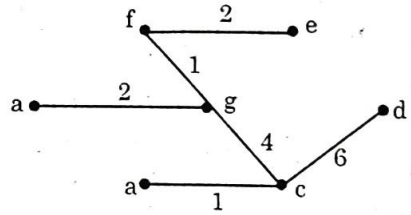


Step–V : In I, VIII, VI, V and III the minimal weight edge is (b, c) = 1.



Step–VI : In I, VII, VI, V & III, the minimal weight edge is (c, d) = 6

[If we take (a, b) = 5, it makes a circuit]



Which is the required minimal weight spanning tree of graph G, with minimal weight

= 1 + 1 + 2 + 2 + 4 + 6 = 16 unit. ■

Matrix Representation of Graphs

A matrix representation of a graph is a very convenient and useful way of graph, such a representation is only possible when the number of vertices and edges are reasonable small.

There are two types of representation of graph into a matrix :

(1) Incidence matrix

(2) Adjacency matrix

Incidence matrix

Let G = (V, E) be a undirected graph with |V| = n and |E| = m and no self loop. Then the incidence matrix of graph G with respect to this ordering of V and E is n × m matrix i.e.

$$A = [a_{ij}]_{n\times m}$$

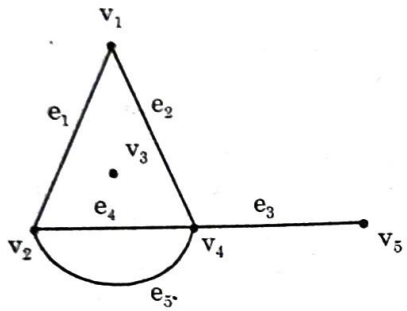
wher $$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident on } e_j \\ 0 & \text{otherwise} \end{cases}$$

It is also written as A(G). Any element of the incidence matrix is either 0 or 1.

Properties of Incidence- Matrix

(1) A row with all zero entries corresponds to an isolated vertex.
(2) A row with a single unit entry corresponds to a pendent vertex.
(3) Since every edge in incident on exactly two vertices, each column of matrix A has exactly two 1's.
(4) Parallel edge in a graph produce identical column in its matrix.

Example :



The incidence matrix of the given graph :

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix :

Let G = (V, E) be a graph with |V| = n and |E| = m and no parallel edges then the adjacency matrix of graph G is an n × n matrix i.e.

$$A = [a_{ij}]_{n \times n}$$

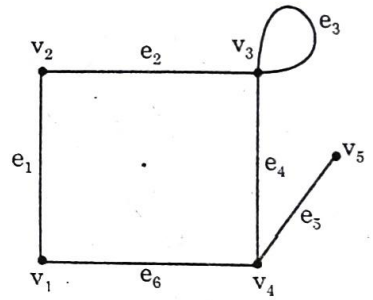
where $$a_{ij} = \begin{cases} 1 & : \text{if } v_i \text{ is adjacent to } v_j \\ 0 & : \text{otherwise} \end{cases}$$

It is a square matrix

Properties of Adjacency Matrix :

(1) Adjacency matrix is symmetric matrix.
(2) All entries of principal diagonal of adjacency matrix all o's if and only if the graph has no self–loop.
(3) If the i^{th} row is the degree of v_i.

Example :



The adjacency matrix of given graph

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Note :

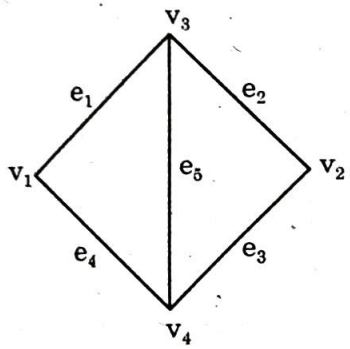
(1) If graph G is disconnected and G_1 and G_2 are components of G then the incidence matrix A(G) of graph G can be given as :

$$A(G) = \begin{bmatrix} A(G_1) & O \\ O & A\,G_2 \end{bmatrix}$$

(2) If graph G is disconnected and G_1 and G_2 are components of the graph G then its adjacency matrix A(G) is given as :

$$A(G) = \left[\begin{array}{c|c} A(G_1) & O \\ \hline O & A(G_2) \end{array}\right]$$

Q. 17. Find incidence matrix and adjacency matrix for the following graph.



Sol. Let G = (V, E) be the given graph with $V = \{v_1, v_2, v_3, v_4\}$ & $E = \{e_1, e_2, e_3, e_4, e_5\}$; $|V| = 4$ and $|E| = 5$

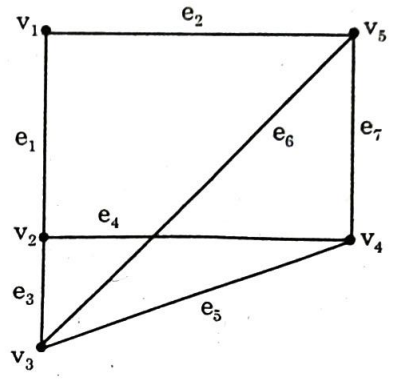
Let A be its incidence matrix then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]_{4\times5} \end{array}$$

Let B be its adjacency matrix then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \left[\begin{array}{cccc} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right]_{4\times4} \end{array}$$

Q. 18. Find the incidence matrix and adjacency matrix of the following graph G :



Sol. Let G = (V, E) be the given graph with $V = \{v_1, v_2, v_3, v_4\}$ and $E = [e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ where $|V| = 4$ and $|E| = 7$

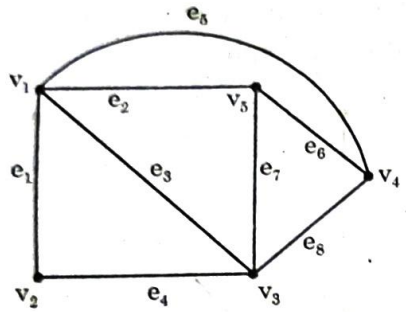
Let A be the incidence matrix of graph G the

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

Let B the adjacency matrix of graph G, then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

Q. 19. Find the incidence matrix and adjacency matrix of the following graph :



Sol. Let G = (V, E) be given graph with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ where $|V| = 5$ and $|E| = 8$

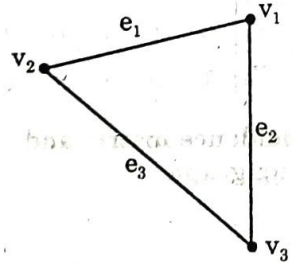
Let A be incidence matrix of G then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let B be the adjacency matrix of G then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q. 20. Find the adjacency matrix and incidence matrix of the following graph :



Sol. Let G = (V, E) be the given graph, with $V = \{v_1, v_2, v_3\}$ and $E = \{e_1, e_2, e_3\}$; where $|V| = 3$ & $|E| = 3$.

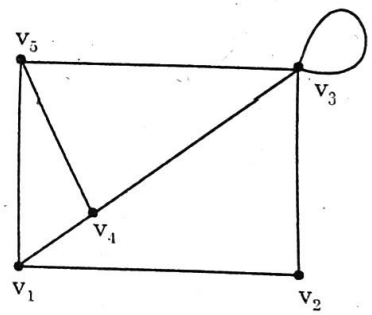
Let A be adjacency matrix of G then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let B be incidence matrix of G then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

Q. 21. Write the adjacency matrix of the following graph :



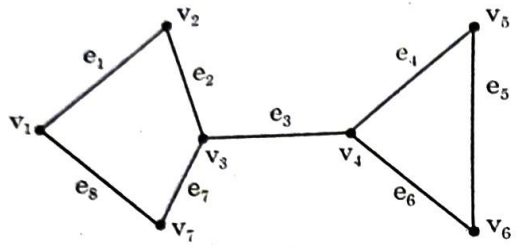
Sol. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ be the set of vertices of given graph G, with $|V| = 5$.

Let A be the adjacency matrix of G then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}_{5\times5} \end{array}$$

■

Q. 22. Determine the adjacency matrix and incidence matrix by the pictures below :



Sol. Let G = (V, E) be given graph with $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ where $|V| = 7$ & $|E| = 8$

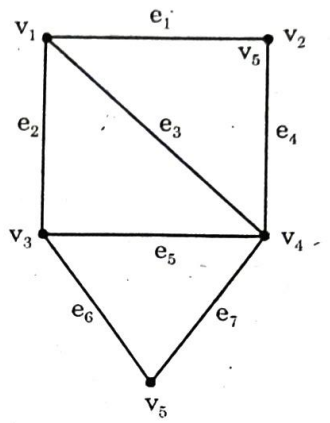
Let A be adjacency matrix of G, then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Let B be incidence matrix of G, then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Q. 23. Find the adjacency matrix and incidence matrix of the following graph :



Sol. Let G = (V, E) be given graph with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

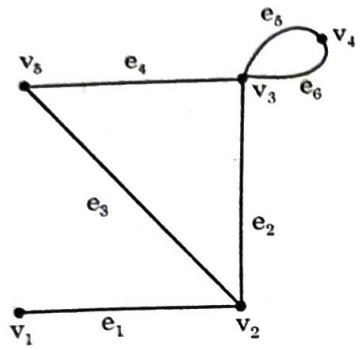
Let A be adjacency matrix of G the

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Let B be incidence matrix of G then

$$B = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

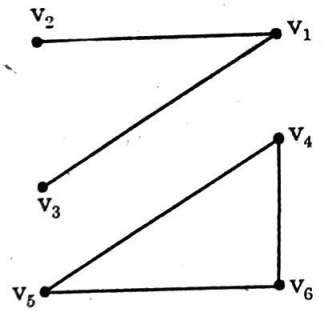
Q. 24. Write the incident matrix of the following graph :



Sol. Let G = (V, E) be given graph and $V = \{v_1, v_2, v_3, v_4, v_5\}$ & $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, Let A be incidence matrix of G, then

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Q. 25. Find the adjacency matrix of following graph :



Sol. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be set of vertices of the given disconnected graph G. Let A be adjacency matrix of G, then.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

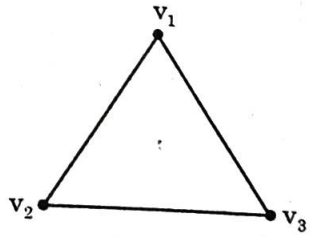
Q. 26. Draw the graph represented by following adjacency matrices :

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$

Sol. (a) Let A be given matrix i.e.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

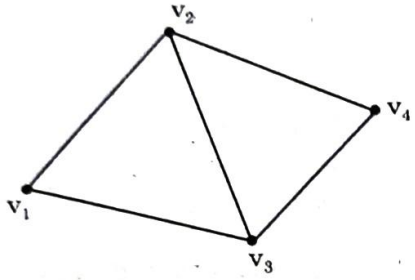
Its graph is :



(b) Let A be given matrix i.e.

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

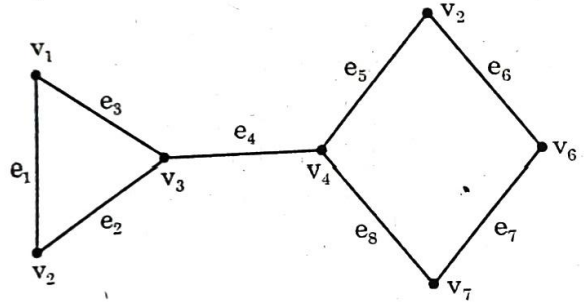
Its graph is



Cut–Set :

Let G = (V, E) be a connected graph. A cut set S of E of G is a minimal set edges whose removal makes G disconnected

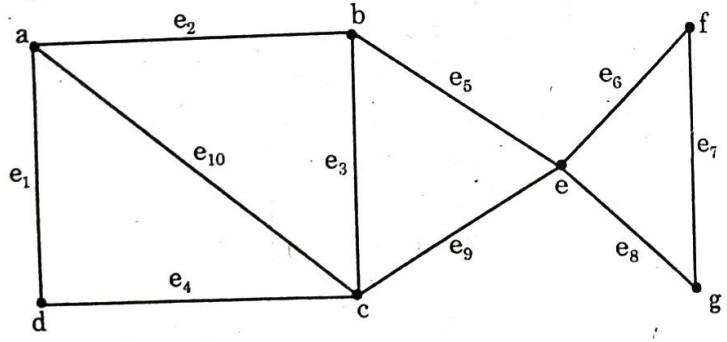
Examples :



Let G = (V, E) be connected graph
where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$
& $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$

Let $S = \{e_4\}$ subset of E is cut set as removal of e_4 will make graph G a disconnected set.

(2)



Let G = (V, E) be a connected graph.
where $V = [a, b, c, d, e, f, g\}$
$E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$
The sets $S_1 = \{e_5, e_9\}$ and $S_2 = \{e_6, e_8\}$ are cut sets. ■

Properties of Cut–Sets

Q. 27. Prove that every cut set in a connected graph G contains at least one branch of every spanning tree.

Sol. Let G = (V, E) be a connected graph and T be a spanning tree in G.

Let S be an arbitrarily cut set in G. Let, if possible, assume that S does not any edge of T, therefore the removal of S from G (i.e. the graph G–S) can not disconnect the graph, since the graph G–S contains the spanning tree T which is a connected subgraph of G.

This is a contradiction to the definition of a cut-set S.

Hence S must contains at least one branch of spanning tree T. ■

Q. 28. Prove that any minimal set of edges containing at least one branch of every spanning tree of a connected graph G is a cut–set.

Sol. Let G = (V, E) be a connected graph, let S be a minimal set of edges which contain at least one branch of every spanning tree of G.

Consider G–S, the subgraph of G after removal of edges of S from G, then the subgraph G–S will not contain any spanning tree of G, since S contains at least one branc of every spanning tree of G.

Therefore G–S is disconnected graph, Thus S is a minimal set of edges which disconnectes G–S. Hence S is a cut–set.

Fundamental Cut–set

Let G be a connected graph and T be a spanning tree of G. A fundamental cut set H of G with respect to T is a cut set of G which contains exactly one branch of T.

Note : Since a spanning tree T of a connected graph with n–vertices has n–1 branches. Therefore, there are n–1 fundamental cut–sets with respect to T.

Cut–Vertex, Bridge and Block

Let G = (V, E) be a graph.

(1) A cut vertex of a graph is that a vertex whose removal increase the number of components.

(2) A bridge of a graph is that an edge whose removal increase the number of components.

(3) A block of a graph is maximal non–separable subgraph of G.

If G is non–separable, then G itself is often called a block.

Planar–Graph

A graph G is said to be planar if it can be drawn on a plane in such a way that no edges cross one another except at common vertices :

A drawing of geometric representation of a graph of any surface with no intersection of its edges is called embedding of the graph.

A graph G that cannot be drawn on a plane without crossing of its edges is called non–planar graph

Examples :

(1)

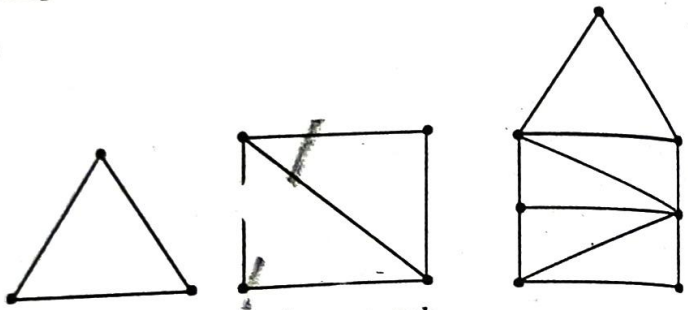


Fig. Some planar graph.

(2)

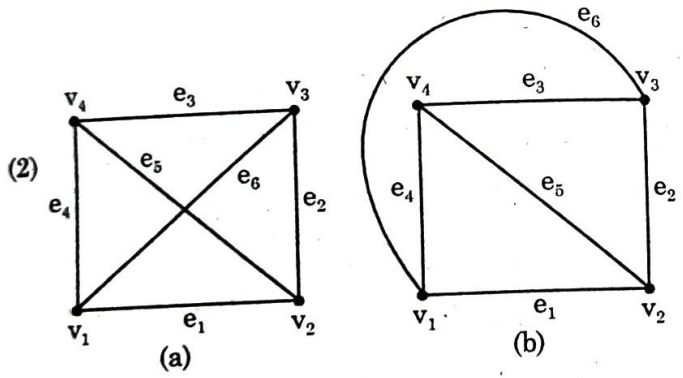
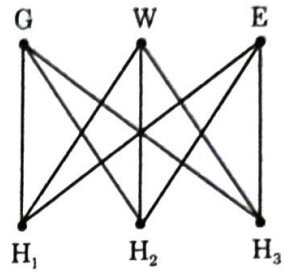
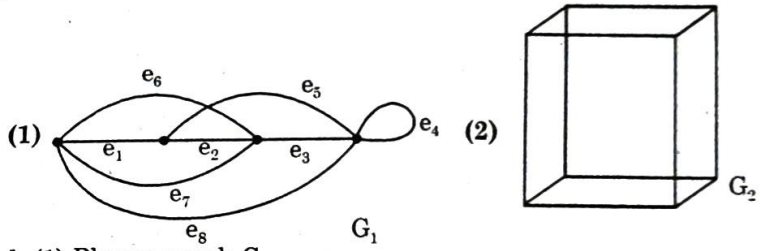


Fig. Graph (b) R graph (c) are embeddings of the graph (a)

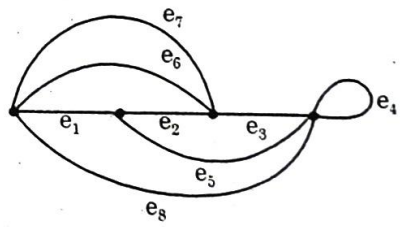


It is not a planar graph i.e. non–planar graph.

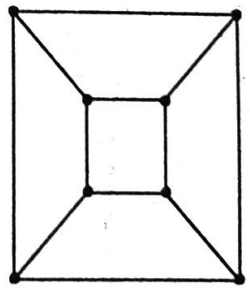
Q. 29. Draw the planar graph of the following graph :



Sol. (1) Planar graph G_1



(2) Planar graph G_2



Map : A particular planar representation of a finite planar graph is called a map.

Region : A planar representation of a graph G divides the plane into various disjoint areas bounded by edges and can not be further subdivided is called the region.

Properties of a planar graph : Let G be a connected planar graph with n–vertices, e–edges and r–regions.

(1) Euler' formula

$$r = e - n + 2$$

(2) $r \leq \frac{2e}{3}$

(3) $e \leq 3n$

(4) $3n - e \geq e$

(5) $\Sigma deg(r) = 2e$.

(6) A complete graph k_n is planar if $n < 5$.

(7) A complete graph $k_{m,n}$ is planar if $m < 3$ or $n < 3$.

Kuratowski's Two Graphs

1. First Kuratowski's Graph

A complete graph with five vertices is known as first Kuratowski's graph.

It is denoted by k_5. It is non–planar graph.

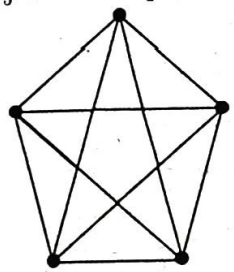


Fig. : First Kuratowski's graph k_5

2. Second Kuratowski's Graph

A regular connected graph with six vertices and nine edges is known as second Kuratowski's graph.

It is denoted by $k_{3,3}$. It is also non–planar graph.

Kuratowski's second graph is also known as utility graph.

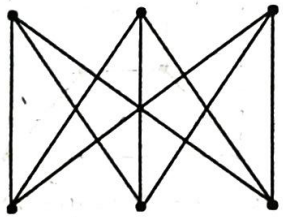
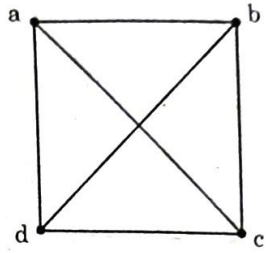


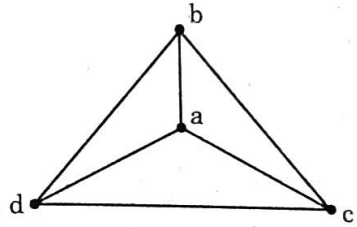
Fig. Second Kuratowski's graph $k_{3,3}$

Q. 35. Show that k_4 is planar.

Sol. Since k_4 is complete graph and drawn as



It can be drawn as

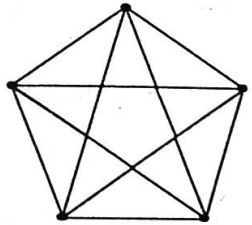


Since this figure is without crossings.

$\Rightarrow k_4$ is a planar graph.

Q.36. Prove that k_5 is not a planar.

Sol. Since k_5 is a complete graph with 5–vertices and 10 edges i.e.



$n = 5$ and $e = 10$

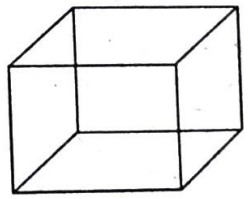
which does not satisfied the condition

$3n - 3 \geq 6$

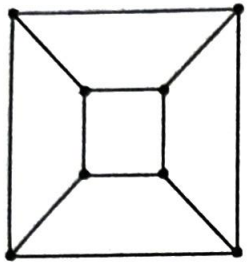
$\Rightarrow k_5$ is not a planar.

Q.37. Prove that Q_3 is a planar.

Sol. Since cuboid Q_3 is drawn as



it can be drawn as



Since this figure is without edge crossing ⇒ it is a planar.

Q.17. Prove that $k_{3,3}$ is non planar.

Sol. We know that $k_{m,n}$ is planar if $m < 3$ or $n < 3$ but $k_{3,3}$ not satisfied the above condition.

$\Rightarrow k_{3,3}$ is not a planar.

Kuratowski's Theorem

A graph is non planar if and only if it contains a subgraph homeomorphic to k_3, k_3 or k_5. ■

Objectives

1. **A connected graph without any circuits is :**
   (a) A path (b) Tree
   (c) Root (d) None of these
2. **In a binary tree the number of vertices are :**
   (a) Even (b) Odd
   (c) Prime (d) None of these
3. **A tree has at least :**
   (a) One vertex (b) Two vertices
   (c) Three vertices (d) None of these
4. **A Leaf in a tree is a vertex of degree :**
   (a) One vertex (b) Two vertices
   (c) Three vertices (d) None of these
5. **A tree with one vertex (no-edge) is called :**
   (a) disconnected tree (b) degenerate tree
   (c) Forest (d) None of these
6. **A tree in which a vertex is of exactly two degree and, rest of the vertices are of one or 3 degree is :**
   (a) A decision tree (b) Rooted tree
   (c) A binary tree (d) A spanning tree

7. **A tree with n-vertices has :**
   (a) n-edges (b) (n – 1) edges
   (c) (n + 1) edges (d) (n – 2) edges
8. **The number n of vertices in a tree is always :**
   (a) Even (b) Odd
   (c) Prime (d) None of these
9. **Number of arms in a spanning tree, generated by a graph with n-vertices :**
   (a) n (b) n – 1 (c) n + 1 (d) 2n
10. **The minimum height of binary tree with 9-vertices is :**
    (a) 2 (b) 3 (c) 4 (d) 5
11. **For 14 vertices tree, min ℓ_{max} =**
    (a) 10 (b) 14 (c) 3 (d) 6
12. **The maximum possible height of a binary tree with 13-vertices :**
    (a) 4 (b) 5 (c) 6 (d) 7
13. **Rank of a graph G is :**
    (a) Number of edges in G (b) Number of chords in G
    (c) Number of branches in any spanning tree of G
    (d) None of these
14. **Nullity of a graph G is :**
    (a) Number of edges in G (b) Number of chords in G
    (c) Number of Branches in any spanning tree of G
    (d) None of these
15. **For a graph G, rank + nullity =**
    (a) Number of edges in G (b) Number of chords in G
    (c) Number of branches in any spanning tree of G
    (d) Number of vertices in G.
16. **A Graph G with n-vertices, (n – 1) edges, and no circuit is :**
    (a) Connected (b) Disconnected
    (c) A network (d) None of these
17. **Every connected graph has at least :**
    (a) One spanning tree (b) Two spannng tree
    (c) Three spanning tree (d) No spanning tree
18. **The tree whose one vertex differs from other is called :**
    (a) Degenerate tree (b) Spanning tree
    (c) Rooted tree (d) None of these

19. A binary tree consists :
(a) Always an odd number of vertices
(b) Always an even number of vertices
(c) Any number of vertices (d) None of these

20. The maximum height of a binary tree with 9–vertices is :
(a) 2 (b) 3 (c) 4 (d) 5

21. The maximum number of paths between each pair of vertices in a tree is :
(a) 1 (b) 2 (c) 0 (d) 3

22. In any graph G = (V, E), with |V| = n, |E| = e and k is number of components, then nullity =
(a) e – n + k (b) e + n + k (c) e + n – k (d) e – n – k

23. In any graph G = (V, E), with |V| = n, |E| = e and k is number of component then rank =
(a) n – k (b) e – k (c) n + k (d) e + k

24. The method to find minimum spanning tree is/are :
(a) Prim's (b) Kruskal's algorithm
(c) Both (a) & (b) (d) None of these

25. Every binary tree is :
(a) Rooted tree (b) Spanning tree
(c) Design tree (d) None of these

26. The incidence matrix of a graph can be drawn if graph is without :
(a) parallel edges (b) Self loop
(c) Both parallel edges and self loop (d) None of these

27. The adjacency matrix of graph can be drawn if graph is without :
(a) Parallel edges (b) Self loop
(c) Both parallel edges and self loop (d) None of these

28. The adjacency matrix is :
(a) Square matrix (b) Symmetric matrix
(c) (a) and (b) both (d) None of these

29. All entries of principal diagonal of adjacency matrix are :
(a) 0 (b) 1 (c) –1 (d) None of these

30. The column sum in an incidence matrix for a simple graph :
(a) |E| (b) |V| (c) 2 (d) > 2

31. Which of the following is not a type of graph :
(a) Eular (b) Hamiltonian (c) Tree (d) Path

32. If a connected graph contains n–vertices, the rank of its incidence matrix is :
(a) n – 1 (b) n (c) n + 1 (d) > n

33. A connected planar graph having 6 vertices and 7 edges, contains regions :
(a) 5 (b) 3 (c) 4 (d) 2

34. The equation of Euler's formula is :
(a) r = e – n + 2 (b) r = n – e + 2
(c) r = e – n – 2 (c) r = n – e – 2

35. How many fundamental cut sets will be generated for a graph with n–vertices :
(a) n(n – 1) (b) n + 1 (c) n(n + 1) (d) n – 1

36. In a complete bipartite graph $k_{m,n}$ the number of edges is :
(a) m + n (b) mn (c) m (d) n

37. Which is the maximum number of edges in a bipartite graph having 10–vertices :
(a) 25 (b) 21 (c) 10 (d) 18

Answer

(1) b	(2) b	(3) a	(4) a	(5) b	(6) c
(7) b	(8) b	(9) b	(10) b	(11) c	(12) c
(13) c	(14) b	(15) a	(16) a	(17) a	(18) c
(19) a	(20) c	(21) a	(22) a	(23) a	(24) c
(25) a	(26) b	(27) a	(28) c	(29) a	(30) c
(31) d	(32) a	(33) b	(34) a	(35) d	(36) b
(37) a					

□□□